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THE GRAVITATIONAL FIELD ENVIRONMENT OF AN EARTH SATELLITE

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SUMMARY

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This report surveys what is currently known about the shape of the earth and the gravitational fields of the earth, the sun, and the moon. The various techniques used to obtain these data are described. A brief survey of the relativistic effects on orbits has also been included.

INTRODUCTION

An earth satellite is subject not only to the earth's gravitational pull but is measurably perturbed by the gravitational influence of the moon and the sun. In this report, which is concerned with the gravitational environment of an earth satellite, it has been necessary, therefore, in addition to discussing the terrestrial gravitational field, to review the entire system of astronomical constants. Moreover, precision determination of orbits and trajectories not only involves knowledge of the gravitational field but also requires knowledge of the shape and dimensions of the earth with high precision so that the relative positions of the tracking stations can be precisely fixed. For this reason the discussion has been extended to embrace the "figure" of the earth.

This report is not simply a recital of values but is an attempt to provide some insight into the various techniques which are used in the determination of these values. As a result of the inevitable delays between preparation of a report of this nature and its final publication, and the bewildering rate of technological advancement, certain of the values quoted herein are already superseded. A case in point is provided by the astronomical unit. Within the last several months, more accurate determinations of the solar parallax have been made by bouncing radar signals off Venus. No point would be served by incorporating this new material, for doubtless even more accurate determinations will be made in the months to come.

The contents of this report fall into the following four main subdivisions:

(a) Shape and gravitational field of the earth using geodetic and gravimetric data,

(b) Utilization of earth satellites for providing data pertaining to the higher harmonics of the earth's gravitational field,

(c) The system of astronomical constants, and

(d) Relativistic effects on orbits.

The last subdivision was included on the grounds that precision of astronomical measurement is such that relativistic effects are beginning to assume significance.

Finally, an appendix has been added in an attempt to introduce the terminology and concepts of spherical harmonic analysis, which is so extensively used in discussing the earth's gravitational field. A bibliography is also included.

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SYMBOLS

$A_e, A_i, A_\Omega, \ldots$ amplitudes of sinusoidal variation of orbital parameters of earth satellite

C moment of inertia of earth about its polar axis

$C_{0,0}, C_{2,0}, C_{3,0}, \ldots$ coefficients of zeroth, second, third, . . . zonal harmonics of the earth's potential field

D coefficient of fourth zonal harmonic as it appears in usual expression for earth's potential (eq. (15))

E total energy (potential plus kinetic)

F force

F_X, F_Y, F_Z force components

G universal gravitational constant

H coefficient of third zonal harmonic as it appears in usual expression for earth's potential (eq. (43))

I moment of inertia

I_X, I_Y, I_Z moments of inertia about principal axes O_X, O_Y, O_Z; O_X, longitudinal axis; O_Y, transverse axis; and O_Z, polar axis

J	coefficient of second zonal harmonic as it appears in usual expression for earth's potential (eq. (15))
K	parameter appearing in second-order approximation to the earth's potential surface
L	lunar inequality in the sun's longitude
M	mean anomaly
N	distance between earth's geoid and the approximating spheroid
P	period of orbital motion
P_e	sidereal period associated with motion of earth-moon system about sun
P_l	sidereal period associated with motion of moon about earth
$P_2^{(0)}, P_3^{(0)}, P_4^{(0)}, \ldots$	second, third, fourth, . . . zonal harmonics
Q	applied moment
R	orbital radius of the earth
$S_1, S_2, S_3, \ldots$	surface harmonics of the first, second, third, . . . degree
T	kinetic energy
U	potential energy
V	deviation potential, $W - U$
W	potential associated with earth's actual gravity field
a	semimajor axis of orbit
b	semiminor axis of orbit
c	velocity of light
d	distance between bodies
e	eccentricity
g	acceleration due to gravity

h height above mean sea level

i orbital inclination

k_s heliocentric gravitational constant (expressed in astronomical units)

l distance between earth and moon

m mass

n mean angular speed of orbital motion

p semilatus rectum or pressure

r radius vector from earth's center

$\bar{s}$ mean radius of the moon

t time

u angular position of satellite relative to its ascending node, $\omega + \theta$

x,y,z Cartesian coordinates

$\beta = \dfrac{g_p - g_q}{g_q}$

ϵ ellipticity of earth spheroid

ϵ' ellipticity of earth's equatorial cross section

η_a parameter appearing in the second-order spheroidal approximation to geoid

θ angular displacement from perigee

κ rotational parameter, $\dfrac{\sigma^2 r_q{}^3}{Gm_e}$

κ' modified rotational parameter, $\dfrac{\sigma^2 \bar{r}^3}{Gm_e}$

λ longitude

λ_1	parameter appearing in the second-order spheroidal approximation to geoid
μ	ratio m_l/m_e
π	mean equatorial horizontal parallax
ρ	density
σ	rotational speed of earth
ϕ	geographic latitude
ϕ'	geocentric latitude
χ	colatitude
ψ	earth's geopotential (superposition of potential and centrifugal force fields)
ω	angle between ascending node and perigee
$\tilde{\omega}$	solar parallax in seconds of arc
Ω	longitude of the ascending node

Subscripts:

b	orbiting body
e	earth
g	due to gravity alone
l	moon
m	mean value
p	value at pole
q	value at equator

CERTAIN BASIC CONCEPTS OF GEODESY

Basic Geodetic Problems

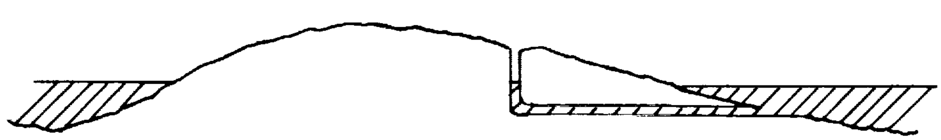
There are two principal problems confronting the theoretical geodesist: determination of the geometrical form of the earth's surface and determination of the earth's gravitational field. It has been said that the earth is rounder and smoother than the average bowling ball; be this as it may, the surface is extremely irregular, consisting as it does of mountain ranges, valleys, ravines, and so forth. When the fine structure of the gravity distribution is viewed, it is also found to be highly irregular. An exact solution to these two basic geodetic problems is beyond the capabilities of practical mathematical analysis. The best that can be done with mathematical analysis is to describe the large-scale undulations of the earth's surface and the large-scale variations of surface gravity.

These factors being considered, it is necessary to define a surface of reference meeting the following two requirements:

(a) That it nowhere departs markedly from the topographical surface.

(b) That it be an equipotential surface to facilitate the analysis of the gravitational field.

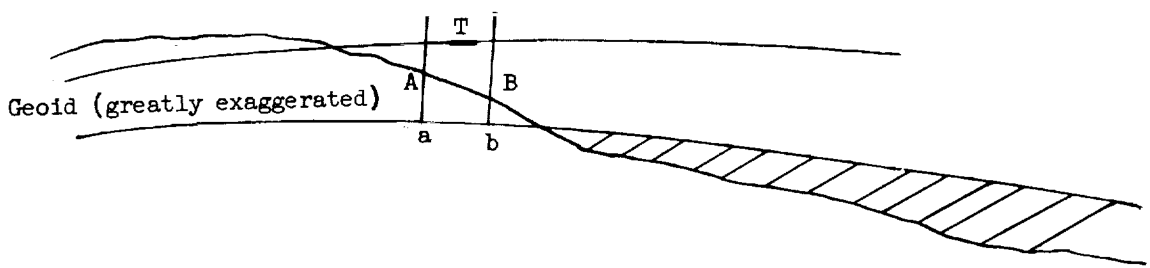
A surface meeting these two requirements which immediately comes to mind is the mean sea-level surface, the so-called "geoid." In addition to meeting these specific requirements, in common with all surfaces of references, it is imperative that it be capable of explicit definition and precise experimental determination. With regard to the definition of mean sea level, some care must be exercised since it is influenced by prevalent trade winds, salt content of water, local temperature, and local barametric pressure; due allowances must be made for these effects. Across continental masses, mean sea level is defined by the conceptual device illustrated in sketch 1.



Sketch 1

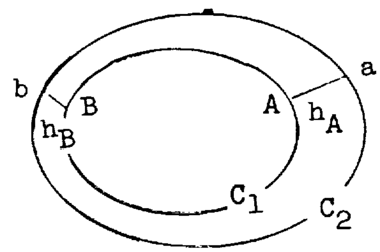
The open ended pipes must be assumed to be infinitesimally small to ensure that the change in mass distribution is entirely negligible.

When its precise experimental determination is considered, a formidable difficulty is encountered. The current technique of spirit leveling does not, as is popularly supposed, give the true height above the mean sea level, that is, above the geoid.



Sketch 2

By starting from a water bench mark and proceeding by highways and railroads, the height differentials between successive points spaced 100 meters or so apart are obtained by the following procedure. Two rods are mounted vertically as at A and B. (See sketch 2.) A telescope is mounted midway between them and exactly level. The difference of reading on rod A and rod B defines the height differential dh between A and B. This is not in all strictness, however, the same as the difference between Aa and Bb, since the telescope lies in the equipotential surface passing through the point T and is not exactly parallel to the portion of the geoidal surface lying directly beneath it. Indeed, the technique of spirit leveling as just described does not even ascribe a unique height $\int dh$ to any point on the surface, the height obtained being dependent upon the route taken between the water's edge and the point in question. This condition results directly from the lack of parallelism between successive equipotential surfaces. The following sketch (sketch 3) and example illustrate this effect and serves to indicate the order of inaccuracy in height estimation.



Sketch 3

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C_1 and C_2 are two equipotential surfaces, C_1 corresponding to mean sea level. Suppose, by using the technique of spirit leveling, the route AabB is traversed and, then a height difference of $h_A - h_B$ between points A and B is assigned in spite of the fact that both points are at sea level.

If the mean gravity between A and a is denoted by g_{m_A}, and the mean gravity between B and b, by g_{m_B}, then

$$g_{m_A} h_A = g_{m_B} h_B \tag{1}$$

$$g_{m_A}\left(h_A - h_B\right) = \left(g_{m_B} - g_{m_A}\right) h_B \tag{2}$$

that is,

$$h_A - h_B = \frac{\Delta g}{g} h \tag{3}$$

However, Δg is of order eg where e is the earth's ellipticity $\frac{1}{297}$ hence

$$h_A - h_B \text{ is of the order of magnitude } eh \tag{4}$$

By taking for h a height of 1 kilometer, which is a representative height of a continental land mass, the discrepancy in h is of the order of 3 meters.

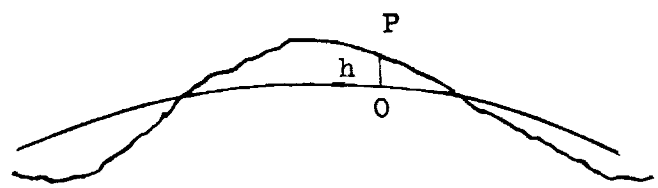
Mathematically this ambiguity in h is expressed in the statement that dh is not a perfect differential. On the other hand, $g\,dh$ is a perfect differential, that is, $\int g\, dh$ taken from sea level to some point P is entirely independent of the route taken in evaluating the integral in question. Indeed, the integral in question defines the potential of P referred to mean sea level as datum. Thus in principle it is possible to determine a unique value of $\psi - C$ for each point on the earth's surface (ψ is the value of the potential at point P, and C is the value of the potential corresponding to mean sea level). Such being the case, a number of unambiguous definitions of height can be formulated, all of which agree to first order with the more crude concept of height. Thus, $\frac{\psi - C}{g_{45}}$ defines the dynamic height where g_{45} is the value assigned to gravity at latitude 45° by the international gravity formula and $\frac{\psi - C}{g_l}$ defines the orthometric height where g_l is the value assigned to gravity at latitude of the observation station on the basis of the same formula, or the height could be defined simply as $\frac{\psi - C}{g}$ where g is the measured gravity at the station without any appeal to the international gravity formula.

The datum surfaces associated with the heights as just defined are referred to as cogeoids. They differ from one another and from the geoid itself by at most a couple or so meters. These cogeoids involving as they do the implicit assumption of the constancy of gravity between a cogeoid and a measuring station are in this regard more compatible with the technique of measuring height by using spirit leveling than is the geoid itself, since the practical determination of height also involves assumption of constancy of gravity or rather constancy of direction of the gravity vector.

An argument which is sometimes advanced against the acceptance of one or other of the cogeoids is that they are not strictly equipotential surfaces whereas the geoid is. This criticism is shown to be lacking in logical foundation when gravity reduction is considered.

Techniques of Reduction

Having introduced the concept of surface of reference, the question now arises as to the method whereby measured values of gravity can be reduced to the surface of reference. Various techniques of reduction exist: (a) free-air reduction, (b) Bouguer reduction, and (c) various isostatic reductions and each technique will be considered in turn.



Sketch 4

Free-air reduction.- In the technique of free-air reduction correction is made only for the difference in distance to the center of the earth without any regard for intervening matter. Thus, by denoting the distance PO by h (see sketch 4),

$$g_P = \frac{Gm_e}{(\bar{r} + h)^2} \tag{5}$$

$$g_0 = \frac{Gm_e}{\bar{r}^2} \tag{6}$$

$$g_0 = g_P\left(\frac{\bar{r} + h}{\bar{r}}\right)^2 = g_P\left(1 + \frac{2h}{\bar{r}}\right) \tag{7}$$

where g_P is measured gravity and g_0 is the value of gravity ascribed to point O.

The value ascribed to gravity at point O is to all intents and purposes the value of gravity which would be measured at O if the overlying mass were compressed onto a surface distribution.

This redistribution of mass implicit in the free-air reduction will result in slight adjustment of the order of a couple of meters to the potential surface. In other words, even though the geoid defines an equipotential surface with the mass where it is and as it is, on applying free-air reduction to gravity it deviates from an equipotential surface by the above amount but this deviation is precisely the order of deviation of the cogeoid from the geoid; hence, the argument favoring adoption of the geoid rather than the cogeoid as a surface of reference, on the grounds that it is truly an equipotential surface, loses its force.

Bouguer reduction.- In this instance, the adjustment of gravity from measuring station P to corresponding point O on reference surface takes into account not only the variation in distance from the earth's center but also the presence of the intervening mass. In this case, the reduced gravity is to a first order the gravity which would be measured at point O if the overlying material were entirely removed. Such a mass redistribution must give rise to a significantly larger perturbation of the equipotential surface than that associated with the free-air reduction.

Isostatic reductions.- Here there is an effective transference of mass to the level of isostatic compensation (100 kilometers). This procedure too gives rise to a greater perturbation to the equipotential surface than does the free-air reduction.

Isostatic reduction has been advocated by some as leading to a smoother and hence statistically more manageable distribution of reduced gravity over the reference surface than does free-air reduction. The extent to which it achieves this purpose is a matter of some contention. (See section on "Astronomical Measurements.") If, as some argue, isostatic compensation only holds to a first order, it is highly debatable whether the advantages compensate for the additional complexity introduced into the analysis (larger distortion of equipotential surface among other things).

It may be wondered why one is quibbling over choice of reference surface when they differ by no more than a couple or so meters. There are two reasons:

(a) It is essential that the basic concepts underlying any physical science be subject to precise definition.

(b) With the development of more sophisticated measuring techniques the experimental inaccuracies may fall below the level of the differences now being discussed in which case they will become significant factors.

Throughout the remainder of this section, the distinctions discussed above will be abandoned and the surface of reference will be referred to as the geoid. The assumption is also made that those topographical features have been compressed into the surface which is tantamount to adopting free-air reduction in preference to isostatic reduction. Furthermore the deviation from equipotential resulting from this redistribution of mass will be disregarded.

ANALYSIS OF EARTH'S FIELD

The Regular Part of the Earth's Field

If the extraneous topographic masses were compressed into the geoidal surface (for practical purposes regarded as the mean sea-level surface), although the earth's contours would then be smooth, it would still not be a perfect sphere. By far, the major deviation from true sphericity is the result of the earth's rotation on its axis. It is logical, therefore, to analyze this dominant deviation first of all, and it is this problem that will be discussed in this section. In a subsequent section the higher order deviations from sphericity will be treated.

At the outset, consider the form assumed by a rotating mass when hydrostatic equilibrium is assumed to prevail, that is, the resultant of the gravitational and centrifugal forces are wholly balanced by hydrostatic pressure forces. The question may well be posed on what grounds the assumption of hydrostatic equilibrium can be justified. There are, of course, differences of opinion as to the extent to which hydrostatic equilibrium prevails in the case of the earth. It is an accepted fact that extraneous masses such as mountain ranges are to some extent compensated for by mass deficiencies in the crustal layers underlying the bases of such features - this is the concept of isostatic compensation. If such stress alleviation occurs in the crustal layers, it is perhaps to be expected that in the deep interior where temperatures are higher, stress alleviation will be even more complete. It is on this basis that the concept of hydrostatic equilibrium is provisionally accepted and with this provision the internal field of the earth is examined.

<u>Equilibrium form of a rotating mass</u>.- If the mass is homogeneous, the problem is amenable to fairly straightforward mathematical solution. Thus, it is readily shown that at low rotational speeds, such as that of the earth, a possible equilibrium form is one in which the surface assumes the form of an ellipsoid of rotation of small ellipticity - the so-called Maclaurin ellipsoids. Moreover, it can be shown that these ellipsoids are stable with respect to small disturbances. It is hardly to be expected, however, that the earth is homogeneous. Indeed, seismological observations show the density at the center to exceed by a factor of about 3 the density of the crustal layer. This lack of homogeneity greatly complicates the mathematical problem.

Under the assumption of hydrostatic equilibrium the Navier-Stokes equations assume the simple form

$$\frac{1}{\rho}\nabla p = \nabla\psi \qquad (8)$$

that is, the gradient of pressure is proportional to the gradient of geopotential. The geopotential field is the result of superposition of the gravitational and centrifugal force fields. This relationship clearly implies that equipotential surfaces are coincident with surfaces of constant pressure and hence surfaces of constant density. There is the additional implication here that, if equilibrium prevails, surfaces of constant pressure are also surfaces of constant temperature.

A first-order analysis assuming hydrostatic equilibrium[1] reveals that the equipotential surfaces, that is, layers of constant density, are ellipsoids of rotation:

$$r = r_q\left(1 - \epsilon \sin^2\phi'\right) \tag{9}$$

where r is the equatorial radius of the layer and ϵ its ellipticity in question. As the center is approached, the ellipticity falls to zero, that is, in the immediate proximity of the center, the equipotential surfaces are to all intents and purposes spherical in form.

Furthermore, the following relationship is established

$$\frac{3}{2}\frac{C}{m_e r_q^2} = 1 - \frac{2}{5}\sqrt{\frac{5}{2}\frac{\kappa}{\epsilon} - 1} \tag{10}$$

where

C	polar moment of inertia
m_e	mass of earth
r_q	equatorial radius of surface layer
ϵ	ellipticity of surface layer
κ	rotational parameter, $\dfrac{\sigma^2 r_q^3}{Gm_e}$

[1]With regard to the question of stability, it seems intuitively obvious that, for the low rotational speeds under consideration, the stable configuration is the one in which the most dense materials have gravitated to the center.

It is of interest to note in passing that from this relation the dependence of ellipticity on rotational speed can be readily established in the case of a homogeneous body $\left(\frac{C}{m_e r_q^2} = \frac{2}{5}\right)$. Substituting into equation (10)

$$\frac{3}{2}\,\frac{2}{5} = 1 - \frac{2}{5}\sqrt{\frac{5}{2}\,\frac{\kappa}{\epsilon} - 1}$$

$$\sqrt{\frac{5}{2}\,\frac{\kappa}{\epsilon} - 1} = 1$$

$$\frac{5}{2}\,\frac{\kappa}{\epsilon} = 2$$

Therefore

$$\epsilon = \frac{5}{4}\,\kappa = \frac{5}{4}\,\frac{\sigma^2 r_q^3}{G m_e}$$

Equation (10) is entirely independent of radial distribution of density. Clearly then, a radial redistribution of density induces changes in both $\frac{C}{m_e r_q^2}$ and ϵ (κ to a first order is not subject to change since σ^2 is fixed) which mutually cancel one another in equation (10). Although this may at first sight occasion some surprise, it becomes somewhat less surprising when it is remembered that the imposition of the weak condition that the surface of an ellipsoid of revolution is to be an equipotential surface imposes certain requirements on the mass distribution in its interior. Specifically,

$$\frac{3}{2}\,\frac{C - A}{m_e r_q^2} = \epsilon - \frac{1}{2}\,\kappa$$

Viewed in this light, it is hardly surprising that the stronger condition of hydrostatic equilibrium leads directly to the relation (10).

These first-order results were first obtained by Clairaut. The analysis has been carried to second order by Callandreau, Darwin, DeSitter, and others. When taken to a second order, the equipotential surfaces assume the form

$$r = r_q\left[1 - \epsilon\,\sin^2\phi' - \left(\frac{3}{8}\,\epsilon^2 + \kappa\right)\sin^2 2\phi'\right] \tag{11}$$

When compared with the equation of an ellipsoid of rotation valid to a second order in ellipticity, that is,

$$r = r_q\left(1 - \epsilon \sin^2\phi' - \frac{3}{8}\epsilon^2\sin^2 2\phi'\right) \tag{12}$$

it is noted that, as a result of the presence of the κ term, the equipotential surfaces are no longer true ellipsoids of revolution. The κ term is itself a function of internal density distribution. By using the density distribution as determined from seismological data, Bullard (1948) assigns a value to κ of 0.68×10^{-6}. By adopting this value of κ and assuming that the equipotential and ellipsoidal surfaces each coincide with the earth's equatorial and polar radii, the maximum deviation between them is found to occur at latitude of 45° and amounts to $\kappa r_q = 0.68 \times 10^{-6} \times 6{,}371{,}000 \approx 4$ meters. Clearly then, from a practical standpoint, even to second order, the equipotential surfaces can permissibly be regarded as spheroidal in form.

The relation (10) carried to second order assumes the more complicated form:

$$\frac{3}{2}\frac{C}{m_e r_q^2} = 1 - \frac{1}{3}\kappa' - \frac{2}{5}\left(1 - \frac{2}{3}\epsilon\right)\frac{\sqrt{1 + \eta_a}}{1 + \lambda_1} \tag{13}$$

where

$$\epsilon'\eta_a = \frac{5}{2}\kappa' - 2\epsilon' + \frac{10}{21}(\kappa')^2 + \frac{4}{7}\epsilon^2 - \frac{6}{7}\epsilon\kappa'$$

$$\epsilon' = \epsilon - \frac{5}{42}\epsilon^2 + \frac{4}{7}\kappa$$

$$\kappa' = \frac{\sigma^2\bar{r}^3}{Gm_e}$$

($\bar{r}$ is now the mean radius of the surface, that is, the radius of a sphere enclosing the same volume as the surface) and η_a and λ_1 are parameters appearing in the derivation which are dependent on the internal constitution of the earth. The relation (13), although no longer wholly independent of internal mass distribution, is quite insensitive to it and provides a basis for what has been regarded as a fairly reliable estimate of ϵ.

It is well that the reader have constantly in mind certain factors that might well contribute to a deviation from hydrostatic equilibrium on which the preceding analysis is based. Some of the more obvious factors are listed below:

(1) The materials of which the earth is composed are not wholly devoid of structural strength.

(2) There is good reason to suppose that density adjustments in the interior are still taking place, that is, heavy elements percolating to the center and lighter elements migrating outwards.

(3) A substantial portion of the earth is still in a molten state. Forced oscillations set up in this liquid core might conceivably be augmented by magnetohydrodynamic dynamo effects.

(4) The axis of figure is not coincident with the axis of rotation.

(5) The earth's rotational speed is subject to slight changes and will give rise to hysteresis effects.

It is likely that factors (4) and (5) are quite negligible. Apart from these, it is difficult to assess quantitatively the magnitudes of the others.

Bearing in mind the uncertainties to which attention has just been drawn, the results of the preceding analysis must be interpreted with some discretion. Fortunately, the analytical result that the earth is to a second order of ellipticity spheroidal in shape is supported by geodesic work; and it is actually on the basis of this empirically determined fact that the development of the theory of the external field of the earth to a second order of ellipticity proceeds.

<u>Theory of external field</u>.- The geoid approximates an ellipsoid of revolution to within a second order of flattening. If the equatorial radius of this ellipsoid of closest fit is denoted by r_q and its ellipticity by ϵ, then

$$r = r_q\left(1 - \epsilon \sin^2\phi' - \frac{3}{8} \epsilon^2 \sin^2 2\phi'\right) \tag{14}$$

If the potential over this surface is assumed to be constant and equal to the value corresponding to the actual earth geoid, then, by Dirichlet's theorem, the external field is uniquely determined. The following expression for the external geopotential field can be derived:

$$\psi(r,\phi') = \frac{Gm_e}{r_q}\left[\frac{r_q}{r} + J\left(\frac{r_q}{r}\right)^3\left(\frac{1}{3} - \sin^2\phi'\right) + \frac{8}{35} D\left(\frac{r_q}{r}\right)^5 P_4(\sin \phi')\right] + \frac{1}{2}\sigma^2 r^2 \cos^2\phi' \tag{15}$$

where

$$\frac{Gm_e}{r_q} = \frac{\psi_o}{1 + \frac{1}{3}\epsilon + \frac{1}{3}\kappa' + \frac{1}{3}\epsilon\kappa' + \frac{2}{15}\epsilon^2}$$

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$$J = \epsilon - \frac{1}{2}\kappa' + \epsilon\left(-\frac{1}{2}\epsilon + \frac{1}{7}\kappa'\right)$$

and

$$D = \frac{7}{2}\epsilon^2 - \frac{5}{2}\epsilon\kappa'$$

Although the external field is uniquely determined, the internal mass distribution of course is not. However, the various harmonics do impose certain requirements on the mass moments of corresponding order without uniquely determining them. Thus, in the general case, there are only five independent harmonics of the second order $P_2^{(0)}$, $P_{2\sin\theta}^{(1)\cos\theta}$, $P_{2\sin 2\theta}^{(2)\cos 2\theta}$ and yet there are six independent second-order moments. In the case under consideration in which there is both axisymmetry and symmetry with respect to the equatorial plane, there are only two independent second-order moments C (polar value) and A (equatorial value). However, only a single requirement is imposed by the second harmonic term, that is,

$$J = \frac{3}{2}\frac{C - A}{m_e r_q^2} = \epsilon - \frac{1}{2}\kappa' + \epsilon\left(-\frac{1}{2}\epsilon + \frac{1}{7}\kappa'\right) \tag{16}$$

Inasmuch as an expression for the external geopotential field has been obtained, an expression for the vector force field can be readily derived. At the surface of the ellipsoid, the effective gravity vectors are normal to it (by virtue of its being an equipotential surface) and the variation of intensity is given by the following expression:

$$g(\phi') = \frac{Gm_e}{r_q^2}\left\{\left[1 + \epsilon - \frac{3}{2}\kappa' + \epsilon\left(\epsilon - \frac{27}{14}\kappa'\right)\right] + \left[\frac{5}{2}\kappa' - \epsilon - \epsilon\left(\epsilon - \frac{39}{14}\kappa'\right)\right]\sin^2\phi' - \frac{1}{8}\epsilon(7\epsilon - 15\kappa')\sin^2 2\phi'\right\}$$

$$= g_q\left[1 + \left(\frac{5}{2}\kappa' - \epsilon + \frac{15}{4}(\kappa')^2 - \frac{17}{14}\epsilon\kappa'\right)\sin^2\phi' - \frac{1}{8}\epsilon(7\epsilon - 15\kappa')\sin^2 2\phi'\right] \quad (17)$$

where

$$\frac{Gm_e}{r_q^2} = \frac{g_q}{1 + \epsilon - \frac{3}{2}\kappa'}$$

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It is convenient to replace the geocentric latitude ϕ' by geographic latitude ϕ defined as the angle which the local normal to the ellipsoid makes with the equatorial plane, since the latter approximates the latitude as experimentally determined by astronomical measurements.

$$r = r_q\left(1 - \epsilon \sin^2\phi + \frac{5}{8}\epsilon^2\sin^2 2\phi\right) \quad (18)$$

$$g = g_q\left[1 + \beta \sin^2\phi - \epsilon\left(\frac{5}{8}\kappa' - \frac{1}{8}\epsilon\right)\sin^2 2\phi\right] \quad (19)$$

where $\left(\beta\right.$ is clearly equal to $\frac{g_p - g_q}{g_q}$ and hence has same meaning in relation to gravity as flattening ϵ has for radius vector $\left.r\right)$

$$\beta = \frac{5}{2}\kappa' - \epsilon - \frac{17}{14}\epsilon\kappa'$$

Measurements of the Earth's Field

Astrogeodetic measurements of the geometric form of the earth's surface yield information pertaining to both its size and shape (that is, r_q and ϵ). Investigations of normal gravity over the surface (gravimetric measurements) permit the determination of both g_q and ϵ. Now $g_q = \frac{Gm_e}{r_q^2}\left(1 + \epsilon - \frac{3}{2}\kappa'\right)$ and, although it is theoretically feasible to use gravimetric measurements for the determination of r_q, the uncertainty associated with Gm_e is so large that it invalidates any estimates of earth's size obtained in this way. Thus, gravimetric measurements provide data bearing on the shape but not on the size of the earth.

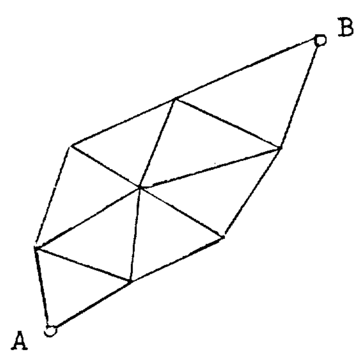
Astronomical measurements consisting of determinations of slight perturbations to which the moon's motion is subject are in essence a measurement of the irregularities in the earth's gravity field and, as a result, are essentially a form of gravimetric measurement. They are therefore subject to the same limitation, that is, they provide data bearing on the shape and not on the size of the earth.

The remainder of this section is devoted to a brief description of astrogeodetic and gravimetric measurements. Such measurements fall within the highly specialized domain of practical geodesy and involve the intricacies of precision measurements and the complexities associated with analyzing statistically masses of data.

Astrogeodetic measurements.- If the earth were truly spherical, its size could be determined by measuring the length of an arc on its surface and the angle subtended by the normals to the sphere at the extremities of the arc in question. The length divided by the angle in radians would serve to fix the radius. If the earth were a true ellipsoid of rotation, two independent arcs would be needed to determine its size and its ellipticity.

From these simple examples it is possible to see that what is involved is the measurement of arcs (the geodetic part of the measurement) and the measurement of angles subtended by normals erected at the extremities of such arcs (the astronomic part of the measurement).

Arcs are not measured directly but by the classical technique of triangulation introduced by Snellius in 1615. The procedure is simple in principle and consists of the measurement of a single base line and the angles of the triangulation net. Thus, the distance between any two points of the net, specifically the distance between points A and B (sketch 5) can be computed.



Sketch 5

Although simple in principle, the procedure is complicated in practice. The chief bugbear confronting the geodesist is the fact that light does not travel in straight lines but is refracted by the earth's atmosphere. In order to minimize such effects in first-order geodetic work, measurement of angles is restricted to the horizontal plane and errors in angle measurement due to refraction are kept down to an arc of about 1/3 second. The upper bound to the length of the side of the triangles comprising the net is thus set by horizontal intervisibility between measuring stations. A lower bound has been set by experience at about 2 miles. Shorter sides would lead to:

(a) Difficulty of defining extremity of arc with sufficient precision. Thus, 1/2 second of arc (which is the precision aimed at in the measurement of angles in first-order geodetic work) corresponds to about 1/3 inch at a distance of 2 miles.

(b) Too rapid accumulation of error.

As it is, base lines are interposed every 10 to 20 triangles to prevent excessive accumulation of error.

With regard to base-line measurement, this is done with Invar tapes which are temperature insensitive and the slight temperature corrections which are applied are therefore extremely accurate. During the making of the measurements, a prescribed tension is applied to the tape either through the use of weights or by means of a spring balance. Moreover, the tape is calibrated in a laboratory before and after each field measurement. Subject to these refinements, it is believed that the measurement of base lines is good to one part in 10^6.[2] It goes almost without saying that base-line measurements must be corrected for height above the geoidal surface. The angles, of course, are not subject to such corrections (unless allowance is made for the fact that the local vertical is not quite alined with the normal to the geoidal surface and such corrections are presumably quite insignificant even to the standard of precision presently under discussion). If the base-line measurement is made at 1 kilometer above the geoidal surface, which is representative of a continental land mass (see sketch 6)

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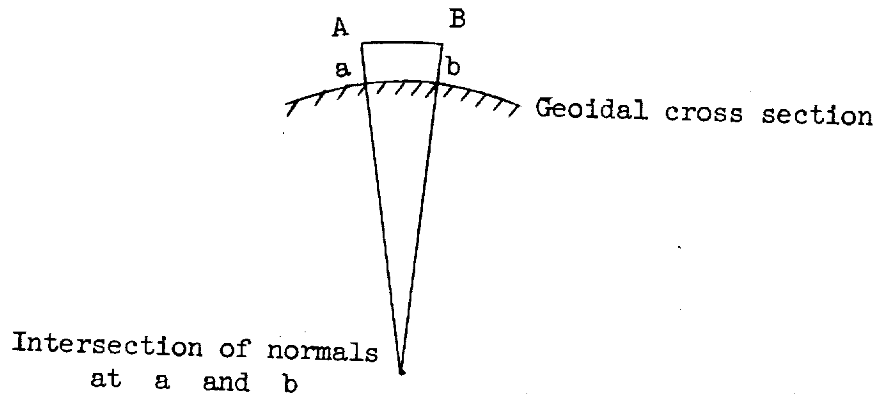
[2]The author is informed that the use of the Invar tape has been largely supplanted by the geodimeter in many countries.



Sketch 6

$$\frac{ab}{AB} = \frac{6378}{6378 + 1} \approx 1 - \frac{1}{6378}$$

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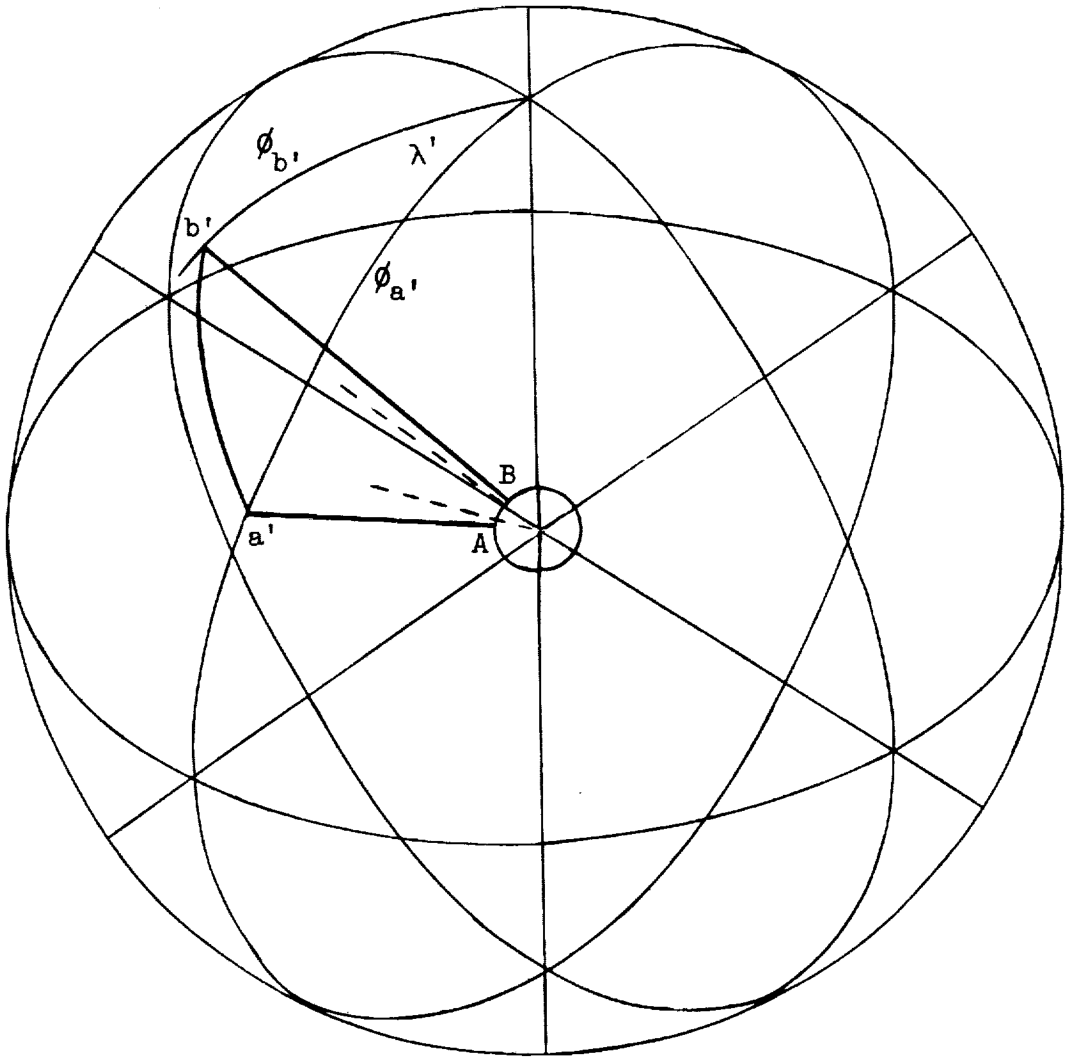
The overall accuracy in geodetic measurement, bearing in mind that inevitable uncertainties involved in individual measurements are reduced by least-square adjustments throughout the triangulation net, is about 0.15 meter per square root of kilometers.

Astronomical measurements.- As the result of the undulations of the geoid, the normals to the geoidal surface at points A and B deviate slightly from the lines drawn through the points in question to the center of mass of the earth. (See sketch 7.) When the differences in local timekeeping are measured, it is the angle λ' which is determined. Astronomical measurement of latitude at A and B determines the angles $\phi_{a'}$ and $\phi_{b'}$. If a'b' is computed and then divided into the measured arc AB, the result is a measure of mean radius of curvature of the geoid. At certain locations this value will exceed the mean radius of curvature of the approximating ellipsoid and at other locations the reverse is true. If these arcs covered the entire earth's surface, these effects would cancel out when a statistical average is taken. Unfortunately, further complications arise. The discussion, as developed thus far, has implied that the local vertical is normal to the geoidal surface. Actually, the local vertical defined by the direction of a plumb bob is influenced by local topographic features, that is, mountain ranges or mass deficiencies or excesses in the crustal layers in the proximity of the observing stations. An attempt is made to allow for such topographic effects by applying some form of adjustment to the measured vertical. Here, two schools of thought are encountered; one favoring isostatic reductions in some form or other, and the other favoring free-air reductions. Those favoring isostatic reductions appear to argue as follows. Since free-air reduction does not adequately represent the influence of



Sketch 7

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topographic features, its application leads to systematic differences between continental masses and oceanic regions, and since moreover astrogeodetic measurements are by their very nature limited to continental regions, the results are correspondingly biased. The proponents of free-air reduction contend, on the other hand, that isostatic compensation is at best an approximation and so-called isostatic reductions are not wholly valid and do indeed introduce systematic errors between land and water regions which exceed those associated with free-air reduction. The issue remains undecided.

The advent of electronic synchronization of time has made it possible to determine longitude with the same precision as latitude and permits the use of both meridional arcs and parallels in the determination of the earth's figure. Prior to this, it had only been possible to use meridional arcs.

Different investigators using different arc systems have obtained somewhat different values for the size and shape of the earth. (See table I.) By and large, the later determinations based as they are on more extensive arc data are more reliable. The determination which is currently favored is the recent determination of Chovitz and Fisher, based on four arcs:

(a) A meridional arc extending from Chile to Canada

(b) A meridional arc extending from South Africa to Scandinavia

(c) A parallel traversing the United States

(d) A parallel extending from Western Europe to Siberia

The values obtained by assuming a flattening[3] $\epsilon = 1/297$ are

(a) Free air, 6,378,240 ± 100 meters

(b) Isostatic, 6,378,285 ± 100 meters

In 1924 the International Astronomical Union adopted the Hayford spheriod (which represented the best determination up to that time)

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[3]Realizing that perhaps more accurate values of flattening are to be obtained by using astronomic techniques, some investigations have assumed a value of ϵ at the outset and used the geodetic data solely to determine the earth's equatorial radius.

as the international ellipsoid. The following figures pertain to this ellipsoid:

$$r_q = 6,378,388 \text{ meters}$$

$$\epsilon = \frac{1}{297.00} = 0.0033670$$

$$\kappa' = 0.00344991$$

$$r = 6,378,388\left(1 - 0.0033670 \sin^2\phi + 0.0000071 \sin^2 2\phi\right) \qquad (20)$$

(where ϕ is the geographic latitude).

Area method in astrogeodetic measurements.- Thus far, the discussion has been centered on arc measurements. There is another technique of astrogeodetic measurements termed the area method which has as its prime objective the determination of the ellipsoid of closest fit to a given region and thus is best adopted to the preparation of maps of the region in question.

It is assumed that the region in question is covered by a two-dimensional triangulation network. A point in the proximity of the center of the net is chosen as a datum point. This point is assumed to lie on an ellipsoid of specified dimensions. Its position on the surface and the orientation of the ellipsoid relative to the triangulation net is fixed if a latitude and longitude is ascribed to the datum point as well as to the azimuth of one of the lines of the net radiating from the point in question. (It would be logical to assign the astronomically determined values to these elements at the outset.)

The size and shape of the ellipsoid and the latitude and longitude of the datum point and azimuth angle comprise the five elements defining the geodetic datum.[4] On the assumption that all points of the network fall on the surface of the reference, their positions can be computed. At a number of preselected points of the network the geographic latitude and longitude (defining the inclination of the normal to the ellipsoidal surface) can be computed. At these same stations the astronomical latitude and longitude is measured and thereby defines the inclination of the normal to the actual geoidal surface. A measure of the closeness of fit

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[4]The geodetic datum used in the U.S.A. has as its initial point Meades Ranch in Kansas, latitude $39^{\circ}13'26''.686$, longitude $98^{\circ}32'30''.506$, azimuth to Waldo $75^{\circ}28'14''.52$. The spheroid used in the triangulation is the Clark spheroid of 1866 (equatorial radius 6,378.206 km; $1/e = 294.98$).

of the ellipsoid to the geoidal surface over the region considered is provided by the relative inclinations of these two normals. Clearly then, the best geodetic datum for the region is obtained by adjusting the five elements constituting the datum in such a way as to minimize the mean square deviations between the ellipsoidal and geoidal normals. Actually the geodetic datum as thus defined is restricted to the extent that it is required to pass through the datum point. If this restriction is relaxed and the height of the datum point above the assumed ellipsoidal surface is regarded as a sixth element of geodetic data, an even closer fit could be obtained. Apparently, geodesists do not think that such ultrarefinements are justified. Frequently in practice not all five elements are necessarily adjusted. Thus, one of the ellipsoids will be selected and the adjustment made solely with respect to the remaining three elements, longitude, latitude, and azimuth. This condition explains the use of the Clarke ellipsoid in the United States of America datum.

Ellipsoids derived in this way will not, in general, be centered on the earth's center and will be wholly inadequate as a reference surface for distant regions. The problem of tying together these various regional geodetic nets is far from complete.

As in most branches of technology, innovations are being introduced into geodesy. Currently, trilateration by radar is the most accurate way of connecting continents. In addition, the capability of shooting flares to high altitudes and putting satellites into orbit permits for the first time the geodetically tying together of the continents by using triangulation techniques.

Gravimetric measurements.- The problem is very simple in principle - it is to determine an expression of the form

$$g = g_q\left[1 + \beta \sin^2\phi - \epsilon\left(\frac{5}{8}\kappa' - \frac{1}{8}\epsilon\right)\sin^2 2\phi\right]$$

which provides the best fit to the observed gravity distribution over the earth's surface. Within recent years, it has been possible to make reasonably accurate measurements of gravity intensity over the ocean masses by using the Vening Meinesz pendulum apparatus. This instrument owes its origin to a desire on the part of Vening Meinesz to make measurements of surface gravity in Holland which happens to be subject to unusually large horizontal displacements. The insensitivity of the apparatus to horizontal displacements enables its use aboard submarines to measure gravity over the oceans (of course, it is necessary that the submarine operate at sufficient depth to avoid vertical motions due to surface waves). Although the precision of such measurements falls somewhat short of the precision of measurements on stable continental masses, the data thus provided form a valuable adjunct to data obtained over land masses and in this regard gravimetric measurements have the edge over astrogeodetic measurements, and the relative significance of free-air reduction as compared with isostatic reduction discussed in relation to astrogeodetic measurements is correspondingly lessened.

Gravity measurements are of two kinds:

(a) Absolute measurements having a precision of five parts in 1,000,000.

(b) Relative measurements having a precision of one part in 1,000,000,000.

These values are given in reference 1 (p. 120). To this precision it is necessary to make due allowance for the attraction of the moon and the sun. Such precision of the relative measurements is particularly surprising when one bears in mind they are made by using the spring balance principle, that is, by observing the extension of a fine silica or Invar wire.

Because of the superior accuracy of relative measurements over absolute measurements, it has become customary to make absolute measurements of gravity only at a number of key stations. Elsewhere, gravity is measured relative to one or other of these key stations. By international agreement the Potsdam value of gravity is accepted as the international datum in all gravity measurements. The initial determination of gravity at Potsdam made in 1906 by Kuhnen and Furtwängler was $981{,}274 \pm 3$ milligals. It is now believed that this value is slightly high as the result of their applying an erroneous correction for support flexure. This error has been confirmed recently by absolute gravity determinations made at National Bureau of Standards in Washington, D.C., and at National Physical Laboratory in Teddington, England. The revised value recommended by Herrick in reference 2 is $981{,}260.9 \pm 0.7$ milligals.

By using the original Potsdam determination as defining the general level of gravity intensity, Heiskanen in 1928 estimated the intensity of equatorial value of gravity at $978{,}049 \pm 1$ milligals. This value, taken in conjunction with the parameters e,m adopted in the definition of the international ellipsoid was proposed by Cassinis and adopted by the International Geodetic Association in 1930 as defining the international formula for normal gravity

$$g = 978{,}049(1 + 0.0052884 \sin^2\phi - 0.0000059 \sin^2 2\phi)$$

(ϕ is geographic latitude). This relationship is based on a flattening $\epsilon = 1/297$.

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From time to time it is to be expected that a slight adjustment will be made to the flattening factor and to the absolute gravity at Potsdam. Both adjustments will contribute a slight change in the value of g_e. Clearly a change in flattening factor will not alter the average of the observed gravity values

$$g = g_q\left(1 + \bar{\beta}\sin^2\phi' + \bar{\gamma}\sin^2 2\phi'\right) \tag{21}$$

(by reverting to geocentric latitude ϕ)

$$g_m = \frac{\sum g}{N} = g_q\left(1 + \bar{\beta}\,\frac{\sum \sin^2\phi'}{N} + \bar{\gamma}\,\frac{\sum \sin^2 2\phi'}{N}\right) \tag{22}$$

If it is assumed that measurements of gravity are made at points uniformly distributed over the earth's surface (such an assumption does not represent too great a departure from the truth; see ref. 2), then

$$\left(\sin^2\phi'\right)_{mean} = \frac{\int_0^{\pi/2} \sin^2\phi' \cos\phi'\, d\phi''}{\int_0^{\pi/2} \cos\phi'\, d\phi'} = \frac{1}{3}$$

$$\left(\sin^2 2\phi'\right)_{mean} = \frac{\int_0^{\pi/2} \sin^2 2\phi' \cos\phi'\, d\phi'}{\int_0^{\pi/2} \cos\phi'\, d\phi'} = \frac{1}{15}$$

Thus

$$g_m = g_q\left(1 + \frac{1}{3}\bar{\beta} + \frac{8}{15}\bar{\gamma}\right) \tag{23}$$

By virtue of the change in ϵ, then g_q, $\bar{\beta}$, and $\bar{\gamma}$ are subject to change but not g_m,

$$\Delta g_m = 0 = \Delta g_q\left(1 + \frac{1}{3}\beta + \frac{8}{15}\gamma\right) + g_q\left(\frac{1}{3}\Delta\beta + \frac{8}{15}\Delta\gamma\right)$$

$$\frac{\Delta g_q}{g_q} \approx -\frac{1}{3}\Delta\beta - \frac{8}{15}\Delta\gamma \tag{24}$$

Since γ is itself second order, adjustments to it can be ignored on the grounds that they will be of even higher order.

$$\beta = \frac{5}{2}\kappa - \epsilon + \frac{15}{4}\kappa^2 - \frac{17}{14}\epsilon\kappa$$

where κ is not subject to change and $\epsilon\kappa$ is second order and its variation can therefore be ignored. Hence

$$\frac{\Delta g_q}{g_q} = \frac{1}{3}\Delta\epsilon \qquad (25)$$

Adjustments to the Potsdam gravity just raise or lower the gravity value as a whole. If this adjustment is denoted by Δg, then

$$g_q = g_{q_0}\left(1 + \Delta g + \frac{1}{3}\Delta\epsilon\right) \qquad (26)$$

This formula serves to determine the adjustment to the equatorial gravity resulting from adjustments in both flattening factor and Potsdam gravity. Some of the gravity formulas derived by different investigators over the past 60 years are given in table II.

As in the case of astrogeodetic measurements the more recent determinations, based as they are on more extensive gravimetric data, are presumably the more reliable.

Sufficient gravimetric data exist to permit the inclusion of longitude terms in the gravity formula. Some attempts to do this are given in table III.

All results predict maximum equatorial gravity in the neighborhood of the longitude of Greenwich and this result can be shown to imply a lengthening of earth's figure in this direction. A number of leading Russian geodists have proposed the following triaxial as a figure of reference.

Mean equatorial radius = 6,378,245 meters

$$\epsilon = \frac{r_q - r_p}{r_q} = \frac{1}{298.3}$$

where r_q is mean equatorial radius and r_p is polar radius.

$$\text{Flattening of equator} \quad \epsilon' = \frac{1}{30,000}$$

$$\text{Longitude of long axis} \quad \lambda_o = 15^o$$

The difference between the ellipsoid of rotation and the triaxial ellipsoid is small. Thus in the case of the Russian ellipsoid, the dimensions of which are presented, the difference between the long and the short equatorial axes is only about 200 meters. Western geodesists view the small advantages to be gained by its adoption as hardly compensating for the additional complications.

Astronomical measurements.- As mentioned previously, this type of measurement involving the observation of the moon's motion can be regarded as a gravimetric measurement made at a remote distance. It has the advantage that, by virtue of its great distance, the higher harmonic irregularities have been smoothed out. Also since the moon is in orbital motion, nature herself, to some extent, does our statistical averaging for us. Such measurements are discussed in more detail in the section concerned with the fundamental astronomical constants.

The Irregular Part of the Earth's Field

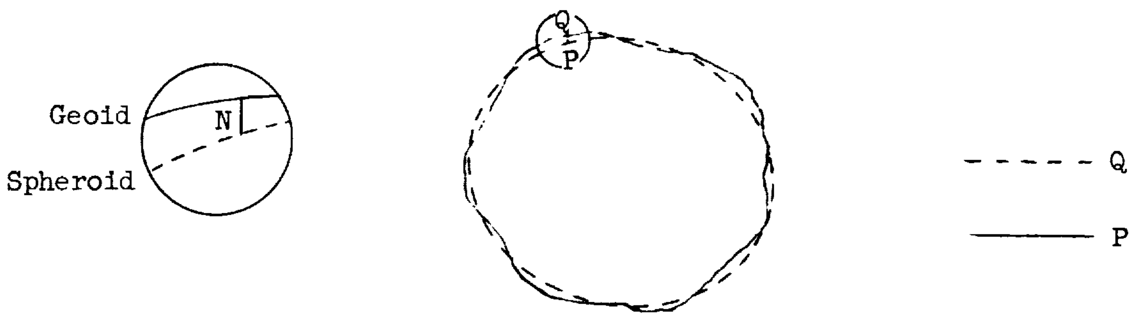
Harmonic analysis.- Up to the present, the earth has been regarded as being spheroidal in form. Deviations from such spheroidal form are indeed relatively minor and it is only within recent years that it has become a matter of practical necessity to determine the extent of deviation. The purpose of this section is to consider the techniques available for the determination of the irregularities of the earth's gravitational field. Let us denote by W the potential associated with the actual earth's field; U has been used to denote the potential associated with the spheroidal earth. The difference is denoted by V; thus,

$$V = W - U$$

and this relation might well be termed the deviation potential. It can be expressed as a series in spherical harmonics as

$$V = \frac{S_2}{r^3} + \frac{S_3}{r^4} + \frac{S_4}{r^5} + \frac{S_5}{r^6} + \ . \ . \ . \tag{27}$$

An attempt will be made to express the deviation of the geoid and the deviation in surface gravity in terms of the deviation potential.



Sketch 8

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Points P and Q (see sketch 8) have the same geocentric coordinates; P, however, is located on the spheroid and Q is located on the geoid. The definition of the potential is such that it represents the work done on a unit test particle by the gravitational field as the particle is brought from infinity up to the point in question. Such being the case, then

$$W_P = W_Q + \bar{g}N$$

where $\bar{g}$ defines the component of the actual gravity force in the direction QP averaged over the segment QP. Hence

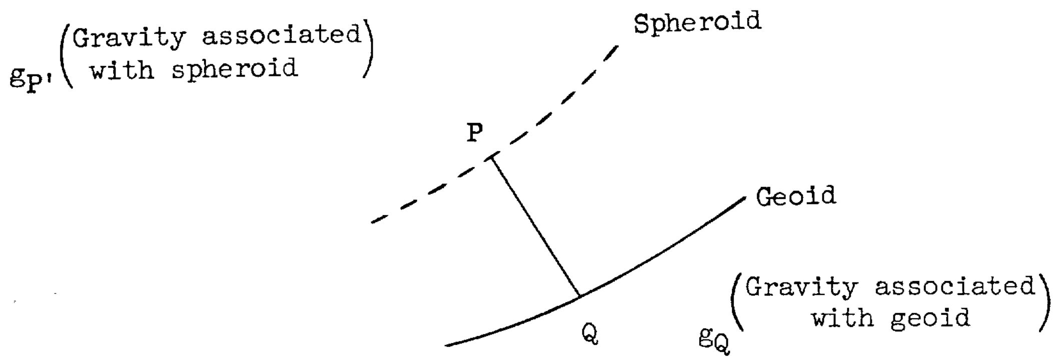
$$W_P - W_Q = \bar{g}N$$

Since the difference $W_P - W_Q$ is a second-order quantity as is indeed the quantity N, it suffices to replace $\bar{g}$ by g where g is now the normal gravity associated with our reference spheroid. In the analysis of the spheroidal earth certain conditions were imposed; one was that the spheroid was to have the same mass as the earth itself and the other was that the potential of the surface of the spheroid referenced to infinity was to be the same as the potential of the actual geoid. For mass distributions which are almost spherically symmetric the potential of an equipotential surface determines the volume enclosed by that surface. In other words, if the masses are rearranged slightly although the equipotential surface itself would be deformed to a slight extent the volume enclosed by that surface would not be subject to change to a first order. It follows that the volume of the spheroid is essentially the same as the volume of the geoid. This being the case, then

$$N = \frac{W_P - U_P}{g} = \frac{V}{g} = \frac{1}{g}\left(\frac{S_2}{r^3} + \frac{S_3}{r^4} + \frac{S_4}{r^5} + \frac{S_5}{r^6} + \cdot \cdot \cdot\right) \tag{28}$$

Such a series development serves to define deviation of the geoid from the spheroid of reference.

The deviation of surface gravity will be considered next.



Sketch 9

The actual gravity at point Q (see sketch 9) is equal to the normal gravity at point P plus the deviation in gravity or the gravity anomaly which is represented by Δg. If the intensity of gravity in the actual potential field is denoted by unprimed quantities and the intensity of gravity in the field of the spheroidal earth by primed quantities, then

$$g_Q = g_{P'} + \Delta g$$

or

$$\Delta g = g_Q - g_{P'} \tag{29}$$

However, by definition

$$g_Q = -\frac{\partial W}{\partial r} = -\frac{\partial U}{\partial r} - \frac{\partial V}{\partial r} = g_{Q'} - \frac{\partial V}{\partial r}$$

Substituting this relation into equation (29) yields

$$\Delta g = g_{Q'} - g_{P'} - \frac{\partial V}{\partial r}$$

However, g' to a first order is equal to $\frac{Gm_e}{r^2}$ and hence

$$g_{Q'} - g_{P'} = - \frac{2Gm_e}{r^3} N = - \frac{2gN}{r}$$

Thus the gravity anomaly is

$$\Delta g = - \frac{2gN}{r} - \frac{\partial V}{\partial r}$$

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and when the expression already derived for N is substituted into this relation

$$\Delta g = - \frac{2V}{r} - \frac{\partial V}{\partial r} \tag{30}$$

Hence, the expression for Δg in the form of a harmonic series is

$$\Delta g = \frac{S_2}{r^4} + \frac{2S_3}{r^5} + \frac{3S_4}{r^6} + \ldots \tag{31}$$

In the light of these series developments for N and Δg, if the gravity anomalies were measured over the entire earth's surface it would be possible in principle to evaluate the harmonic terms of series (31) and hence calculate the form of the geoid over the entire globe, ocean masses as well as land masses. Thus by gravimetric analysis it would be possible to establish the actual form of the geoid which is something that could not be done by astrogeodetic measurements.

Stokes formula.- In a wide variety of problems in applied mathematics whether they be problems in dynamics of vibrating systems, problems in elasticity, or problems in heat conduction, there are two approaches open; one is the approach of determining the response to a sinusoidal fluctuation and the other is that of determining the response to a unit impulse (the Fourier approach or Green's approach, respectively). So it is in the present problem. Thus a unit gravity anomaly, that is, the gravity anomaly which at one point P is equal to unity and everywhere else is zero can be considered. By using the series developments already derived, it is then possible to compute the form of the geoid associated with such a gravity distribution. It is found to be of the form

$$N(\theta,\phi) = \frac{F_o}{4\pi gr}$$

where

$$F_o = \csc \frac{1}{2}\psi + 1 - 5 \cos \psi - 6 \sin \frac{1}{2}\psi - 3 \cos \psi \log_e\left[\sin \frac{1}{2}\psi\left(1 + \sin \frac{1}{2}\psi\right)\right]$$

and ψ is the angle between the direction of θ,ϕ and θ_P,ϕ_P. In the general case, therefore, in which the gravity anomaly is Δg arbitrarily distributed over the entire earth's surface, the deviation of the geoid is then given by the integral expression

$$N = \frac{1}{4\pi gr} \int F_o \, \Delta g \, dS \tag{32}$$

where F_o is clearly the influence function or the Green's function appropriate to the problem. This formula was first derived by Stokes. At first sight it appears to offer no particular advantage over the harmonic approach which has been described previously, since it also necessitates knowledge of Δg the gravity anomaly over the entire earth's surface. Circumstances, however, can be envisaged in which it would be definitely advantageous. Thus, suppose that the deviation of the geoid at any particular point is insensitive to gravity anomalies over the more distant portions of the global surface. In this instance the integration over a neighborhood could be limited in the immediate proximity of the point P. Heiskanen and Vening Meinesz (ref. 1) have suggested that such is indeed the case. Since they regard the deviations from equilibrium as being relatively minor and moreover relatively localized, they suggest as an upper limit to the deviation of gravity anomaly, 30 milligals megameter^2. If the deviations of the center of such a gravity anomaly are computed by using Stokes' formula they are found to be of the order of 20 meters and external to the circle 3,000 kilometers radius the deviation is of the order of 1.00 meter. This being the case, they have calculated that, if Stokes' formula is used to calculate the deviation of the geoid at the center of an area over which gravimetric data are available to a radius of 3,000 kilometers, ignorance of the gravity anomaly in the more distant regions is not likely to lead to an error in excess of more than 2.6 meters which is quite insignificant. Jeffreys (ref. 3), on the other hand, holds a radically different view as to the extent of the earth's deviation from equilibrium. He has suggested that there are very wide-scale undulations in the earth's gravitational field with amplitudes up to 100 meters and, if he is to be believed, then Stokes' formula is only applicable if the gravity measurements are available over the entire earth's surface. Within recent months, analysis of orbital data has decided the issue in favor of Jeffreys. (See section "Geophysical Implications.")

USE OF EARTH SATELLITES FOR THE DETERMINATION OF THE EARTH'S GRAVITATIONAL FIELD

Nature of the Problem

The motion of an earth satellite is modified by the successive harmonics of the earth's gravitational field and by observing and measuring these perturbations one can deduce the magnitudes of these successive harmonic terms. In addition, there are a number of extraneous perturbational influences and although these complicate the analysis of the earth's gravitational field, if a broader viewpoint is taken these additional perturbations are to be welcomed rather than deprecated, inasmuch as they themselves provide data of very geniune scientific interest. Among these extraneous perturbational influences are (a) air drag, (b) electromagnetic drag, (c) lunisolar perturbations, and (d) solar radiation pressure; influences (a), (c), and (d) have been proven to be of significance and are detectable in the case of the Vanguard orbit. There is some controversy with regard to the significance of influence (b). There are other perturbational influences, one such influence being the tidal influence. Although this influence has been examined theoretically, no one would claim that this is other than of academic interest at the moment. A short discussion of these extraneous perturbational influences is included because not only do they have intrinsic interest in themselves but in addition they have direct bearing on our analysis of the earth's gravitational field, inasmuch as they will establish, by virtue of the irregularities to which they are subject, the gravitational noise level which will set a limit to the extent to which the harmonic analysis can be carried and will also determine the feasibility of detecting some of the very minute relativistic effects. One of the requirements imposed on the orbit of a geodetic satellite is that its perigee distance not be too great. As a result therefore it is found that, of all the extraneous perturbational influences, air drag is indeed the dominant one. It is logical therefore that this influence is the one that is considered first.

Discussion of Extraneous Perturbations

<u>Perturbations associated with air drag</u>.- Air drag forces are operative at each point of the satellite's orbit; however, they are clearly maximized in the neighborhood of perigee where the air is densest and where the velocity of the satellite is greatest. Indeed for orbits having eccentricity in excess of 0.02 virtually the entire drag contribution comes from the immediate proximity of perigee and the air drag can be regarded as inducing a succession of retarding impulses as the satellite passes through perigee. This condition will give rise to a

progressive shrinking of the satellite orbit with an associated diminution in periodic time and a decrease in eccentricity. The changes wrought by the earth's gravitational field are of an entirely different nature consisting as they do of progressive movements of the node and perigee with superimposed periodic variations in all the orbital parameters. Thus, the task of separating the influence of air drag from that of the gravitational effects is simplified. Thus, it may be said that the coupling between air drag on the one hand and gravitational field on the other is relatively minor.

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Jacchia[5] noticed that superimposed on the progressive change in the satellite's orbital period was an oscillation of 27-day periodicity. (See sketches 10 to 12 which are taken from ref. 5.) Some argued that such a periodicity was in some way associated with the moon's motion, perhaps the result of the perigee passing through the lunar tidal bulge. The latter contention was clearly untenable since the moon produces not one tidal bulge but two; one at the sublunar point and the other at the point diametrically opposite. The most plausible explanation to date has been advanced independently by Jacchia (ref. 5) and King-Hele (ref. 6). They associate the 27-day periodicity with the sun's rotation on its axis. For some time it has been known that the magnetic field of the earth is subject to a 27-day periodicity. This periodicity has been attributed to the existence of activity centers on the surface of the sun. These activity centers may persist for a period of several months. From them it is believed particles are continuously ejected. Each revolution the earth is sprayed by these particles and this spray gives rise to the variation in the earth's magnetic field and might equally well be expected to give rise to a change in the density of the atmosphere as a whole. Since the 27-day periodicity was particularly marked in the case of Vanguard I which happened to have the highest perigee, it appears that the density change is proportionately greater at higher altitudes. Such an explanation implying as it does that the atmosphere as a whole is subject to change is borne out by the observation that if the orbital data of Sputnik II, Sputnik III, and Vanguard I are compared, the peaks and hollows occur roughly in the same location. Further evidence in support of this contention was not long in forthcoming; thus, Jacchia noticed the orbital period of Sputnik III was subject to particularly abrupt changes in July and December of 1958 (sketch 13 from ref. 7), in each instance, about 24 hours after the onset of a major solar flare. No such change was noticed, however, in the case of Vanguard I. It is known that the flares give rise to plasma clouds and it is further known that it takes approximately 24 hours for these plasma clouds to reach the proximity of the earth. In this instance, it is believed that the plasma cloud on reaching the earth filled the outer part of the Van Allen

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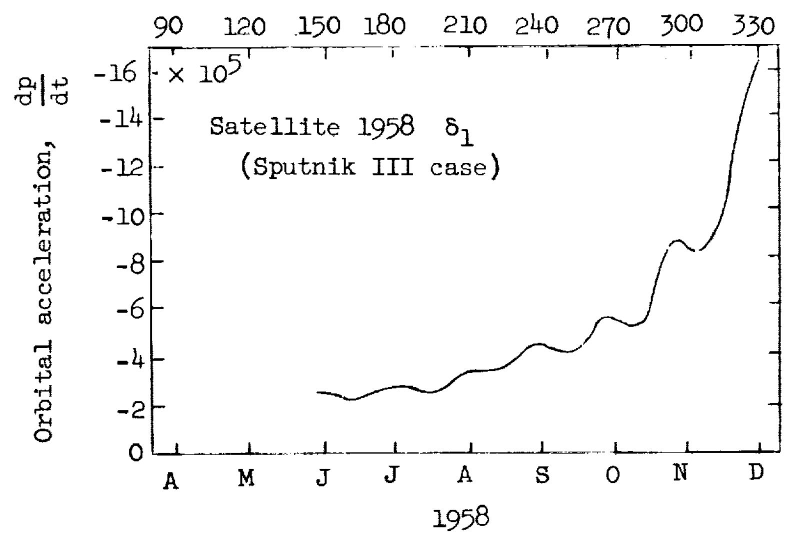
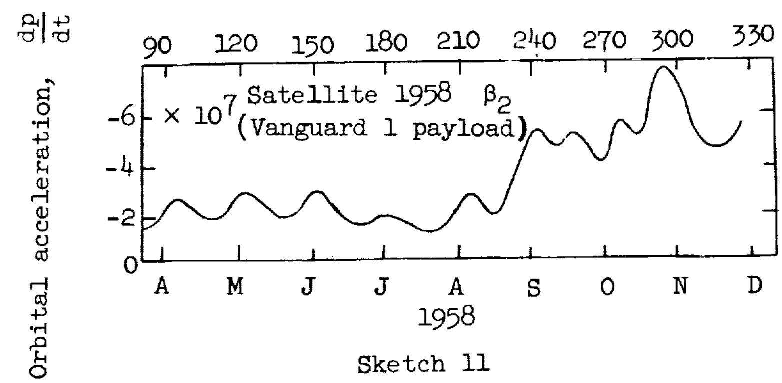
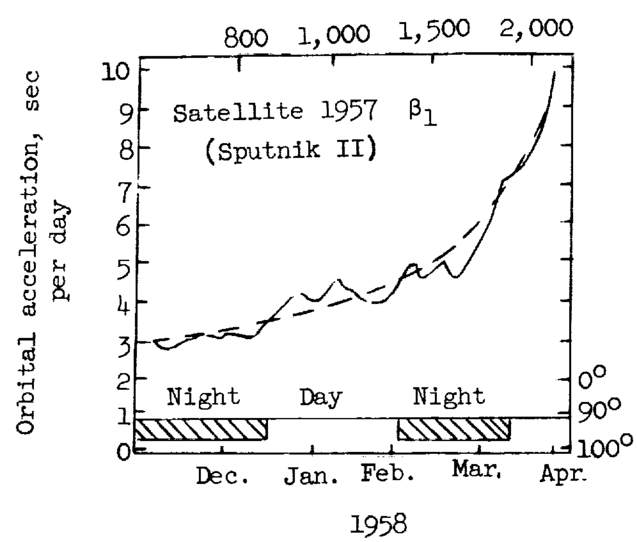
[5]Harvard College Observatory Announcement Cards 1391 and 1392. (See ref. 4, p. 345.)



Sketch 10



Sketch 11



Sketch 12

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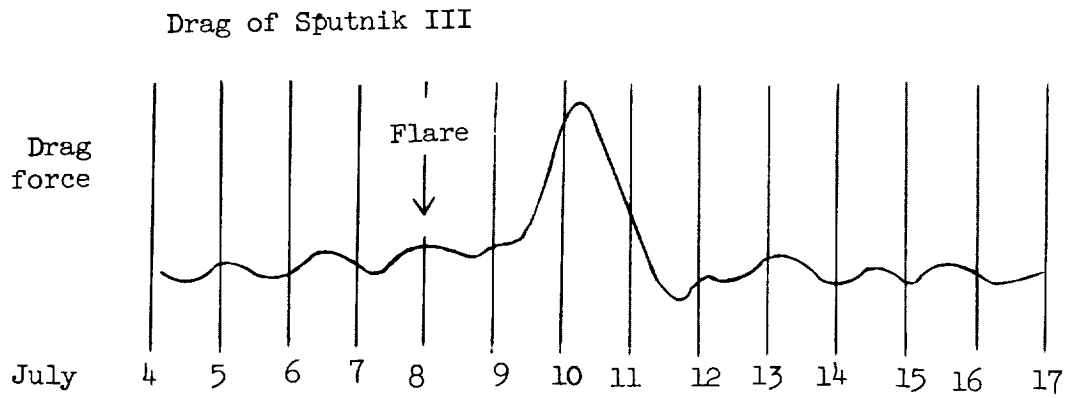
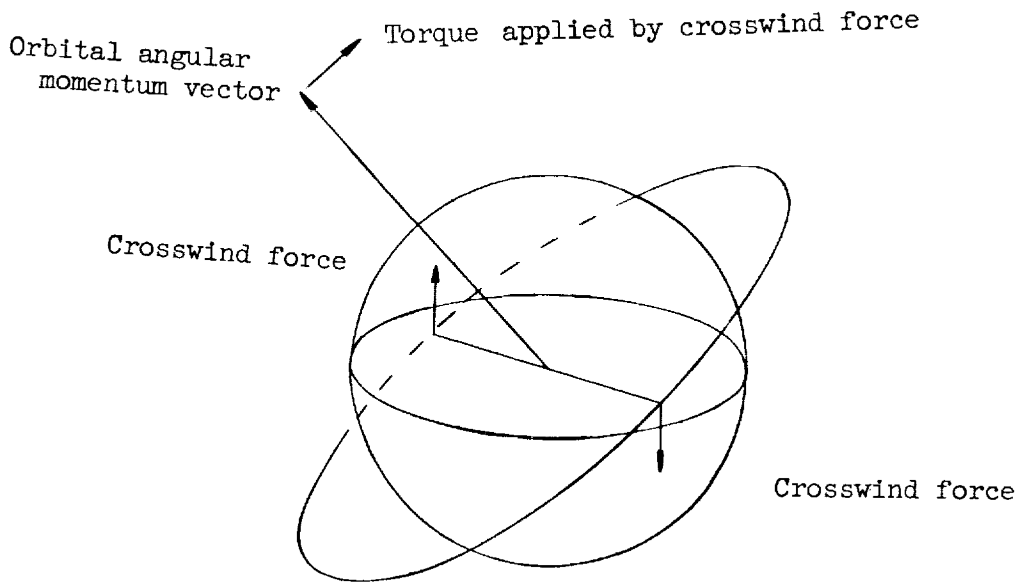


Sketch 13

radiation belt. Particles overflowed through the polar extremities of this belt, heated the air in the auroral zones, and produced an increase in density. The orbit of Sputnik III happened to pass through the auroral zone and this condition would account for the augmented drag. For Vanguard I, on the other hand, which had a lower orbital inclination, the effect did not manifest itself. There are, however, still certain features awaiting explanation; one such feature is the fact that the 27-day periodicity in the geomagnetic field is only detectable during periods of solar minimum, whereas the 27-day periodicity in the orbital period of the earth's satellites has, however, been observed during the past period of solar maximum.

In addition to the 27-day periodicity, an examination of the decrease in the orbital period disclosed a periodicity of somewhat longer duration. In sketch 12 those intervals during which the perigee is on the night side of the earth and those intervals when the perigee is on the daylight side have been defined. The drag level of the earth is higher when the perigee is in daylight than when it is in darkness. This result is to be expected since the air on the day side is warmer and hence the density at any particular altitude is greater. The same effect is to be noted in the case of Vanguard I; thus, there is an increase in a drag level as the perigee moves through the proximity of the subsolar point. (See sketch 11.)

In addition to these periodic variations in drag level, air drag also induces a change in the orbital inclination. This effect is only manifest during the final stages of the satellite's orbital decay when the satellite is moving through relatively dense air. The earth's atmospheric mantle participates, of course, in the earth's rotation and this participation induces cross winds which will result in a decrease of the inclination of the satellite's orbit (sketch 14) as is observed. The assumption is that the satellite is moving in a counterclockwise direction when viewed from the north. If the satellite were moving in the reverse direction, then the orbital inclination would be expected to increase.



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Sketch 14

Jastrow (ref. 7), although agreeing that the observed variations in drag intensity are attributable to solar activity, has suggested that these effects may well not be the result of a direct increase in air density but rather the result of an augmentation of ionization level which in turn will increase the electromagnetic drag forces. On this question, there appears to be a difference of opinion and this seems to be an opportune moment in which to introduce a brief discussion of electromagnetic perturbation forces.

<u>Electromagnetic perturbation</u>.- An earth satellite, by and large, moves in an ionized medium, the ionization level being particularly high within the radiation belts. The temperature of the medium is such that the speed of the ions is appreciably less than the satellite speed. The speeds of the electrons, on the other hand, are appreciably greater. As a result, the ions are encountered only by head-on impact; the electrons, by virtue of their great mobility, impact the satellite from all directions. As a consequence, there are a much greater number of impacts of

electrons than there are of ions and, as a result, the satellite acquires a negative charge. Jastrow and Pearse (refs. 7 and 8) were the first to examine this effect and estimated that a voltage of the order of 1,000 volts might well be acquired during passage through the inner radiation belt. More recent and more detailed analyses have suggested that the original estimate of Jastrow and Pearse grossly overestimated the magnitude of the effect and that the negative voltage acquired would be more likely to be of the order of one-quarter of a volt.

The charged satellite moving through the earth's magnetic field would be subjected to a perturbing force and its orbit would be to some extent modified. The magnitude of the effect can be gleaned from reference 9, in which a satellite of 0.8 pound mass is moving in a circular orbit with a radius of 6.5×10^3 kilometers within the earth's magnetic equatorial plane. The satellite in question suddenly acquires a charge of 10^{-3} abcoulombs and it is calculated that the satellite during one circuit will be perturbed to the extent of approximately 20 kilometers. On the basis of these numbers, one might adduce that the effect is a very significant one indeed. However, when the matter is considered in a little more detail, a mass of 0.8 pound, if the satellite is assumed to be a balloon satellite composed of Mylar, would have a diameter of about 5 meters and for such a sphere to hold a charge of 10^{-3} abcoulombs implies that the voltage would necessarily be of the order of 40×10^6 volts. Clearly then on introducing more realistic numbers pertaining to the ratio of mass to charge the orbital deviation is going to be of the order of millimeters rather than kilometers and the effect is going to be quite undetectable.

There is a further effect to which a charged satellite is subjected. A satellite having a negative charge becomes surrounded by a positive space charge and hence the cross section relative to ion impact (and it is these ion impacts which determine the intensity of the electromagnetic drag) is increased. Jastrow and Pearse estimated that the increase in cross section may be a factor of tenfold to fiftyfold. However, it is seen that Jastrow and Pearse did indeed grossly overestimate the magnitude of the effect in question and the present belief is that the effect is very small. For this reason, the consensus is that variations in drag intensity observed in the satellite's motion are to be attributed to air drag rather than to electromagnetic drag.

Lunisolar perturbations.- A detailed examination of the trajectories of Vanguard I disclosed a periodic variation in perigee height of about 1 to 2 kilometers with a periodicity of 449 days. Such a variation in perigee could not, of course, be attributed to drag forces nor could the periodicity be associated with any of the harmonics of the earth's gravitational field. It was Kozai[6] who first determined the source of this

[6] Kozai: New York Times, Aug. 21, 1959.

perturbation as the lunisolar influence. Such perturbations can become very appreciable if resonances occur. Thus, if the perigee moves in step with the sun or the moon, there may result a progressive change in perigee of the order of 1 kilometer per day over a period of several years. Clearly, due allowance must be made for this perturbation in computations of the lifetime of earth satellites.

Solar radiation pressure.- A further perturbation which was discerned in the case of Vanguard I was a variation in perigee height with a period
of 850 days. Such a periodicity corresponded to the diurnal motion of L
perigee, that is, the time taken for the perigee to make one complete 1
rotation relative to the sun. It seems reasonable to suppose that this 9
perturbation is the result of solar radiation pressure. It may occasion 7
some surprise that radiation pressure forces induce changes in orbit com- 9
parable to the lunisolar perturbations since the radiation pressure forces are only about 10^{-5} of the gravitational forces. However, one must bear in mind that the solar gravitational forces influence the earth and the satellite almost to the same extent and the gravitational effect is the difference between two large numbers whereas in the case of radiation pressure this condition influences the satellite orbit alone. This perturbational influence also must be considered in computations of the lifetimes of earth satellites. Therefore, the magnitude of the solar radiation perturbation increases as the ratio of satellite area to satellite mass increases and thus satellites such as Echo will be particularly susceptible to perturbations of this kind.

Detailed Analysis of the Earth's Gravitational Field

The presentation in this section follows closely, albeit superficially, the treatment as given in reference 10. The zonal harmonics by virtue of their axisymmetry do not introduce time dependency into the gravitational field as a result of the earth's rotation. The tesseral harmonics do. It is intuitively obvious, however, that the influences of these tesseral harmonics will be effectively smoothed out in the case of Vanguard and other satellites which have been used thus far for geodetic purposes and in order to detect the tesseral harmonics it would be necessary to use a satellite having an orbital period of approximately 24 hours. Any changes in periodic time therefore must be attributed to the effect of air drag. From the observed changes in periodic time one can compute the associated changes in the semimajor axis. Hence, since drag does not produce any change in perigee distance, the variation in eccentricity resulting from air drag can be computed. Denote the drag contribution to eccentricity by Δe, denote the measured value of the eccentricity by e_{meas}, and the eccentricity which would have been measured had drag forces not been operative by e_g. Then

$$e_g = e_{meas} + \Delta e$$

From an empirical analysis of the orbital data

$$e_g = e_o + A_e \sin \omega \tag{33}$$

The orbital inclination i itself is not significantly influenced by drag except in the final stages of orbital decay. Thus,

$$i_g = i_{meas} = i_o + A_i \sin \omega \tag{34}$$

The influence of air drag on the location of the nodal point and the perigee position are somewhat more complicated. Thus, the rate of nodal procession $\dot{\Omega}$ and the rate of perigee movement $\dot{\omega}$ are given by

$$\dot{\Omega} = - \frac{3}{2} \frac{n \cos i}{p^2 Gm_e} C_{2,0} \tag{35}$$

and

$$\dot{\omega} = \frac{3n}{p^2 Gm_e}\left(1 - \frac{5}{4} \sin^2 i\right) C_{2,0} \tag{36}$$

It will be noted that these rates are not only dependent upon the oblateness of the earth but are also influenced by the semimajor axis and the orbital eccentricity and, since these parameters are in turn influenced by drag, drag will in turn modify to some extent both $\dot{\Omega}$ and $\dot{\omega}$. If the change in the semimajor axis induced by drag during the interval, epoch up to time τ, is denoted by δa then the drag contribution to nodal procession at time τ is given by

$$\dot{\Omega} = - \frac{3 \cos i}{4Gm_e a^{9/2}\left(1 - e^2\right)^2} C_{2,0}\left(\frac{7 - e}{1 + e}\right)\delta a$$

The drag-induced change in the nodal location after time t is therefore

$$\Delta\Omega = \int_{t_0}^{t} (\delta\dot{\Omega}) d\tau = - \frac{3 \cos i}{4Gm_e a^{9/2}\left(1 - e^2\right)^2} C_{2,0}\left(\frac{7 - e}{1 + e}\right)\int_{t_0}^{t} (\delta a) d\tau$$

By the same token

$$\Delta\omega = \int_{t_0}^{t} (\delta\dot{\omega}) d\tau = \frac{3\left(4 - 5 \sin^2 i\right)}{8Gm_e a^{9/2}\left(1 - e^2\right)^2} C_{2,0}\left(\frac{7 - e}{1 + e}\right)\int_{t_0}^{t} (\delta a) d\tau$$

Thus, the following expressions are obtained

$$\Omega_g = \Omega_m + \Delta\Omega = \Omega_0 + \dot{\Omega}_0(t - t_0) + A_\Omega \cos \omega \tag{37}$$

and

$$\omega_g = \omega_m + \Delta\omega = \omega_0 + \dot{\omega}_0(t - t_0) + A_\omega \cos \omega \tag{38}$$

Equations (33), (34), (37), and (38) serve to define the time dependence of the orbital parameters resulting from gravitational field alone. Thus, the influence of air drag is effectively eliminated. Denote the earth's gravitational field by the following series:

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$$W = \frac{C_{0,0}}{r} + \frac{C_{2,0}}{r^3} P_2{}^{(0)} + \frac{C_{3,0}}{r^4} P_3{}^{(0)} + \frac{C_{4,0}}{r^5} P_4{}^{(0)} \tag{39}$$

The tesseral harmonics are omitted for reasons already given. The first harmonic is omitted by virtue of the fact that the origin of the coordinate system of reference is placed at the center of the earth. The second and fourth harmonics contribute to the progressive movement of the node and perigee, that is, they contribute to $\dot{\Omega}_0$ and $\dot{\omega}_0$ and the measured values of these parameters can be used to deduce the magnitude of both the second and fourth harmonics. In this way, the following values are obtained:

$$C_{2,0} = -17.55 \pm 0.001 \text{ (megameters)}^5\text{(kiloseconds)}^{-2}$$

$$C_{4,0} = 1.12 \pm 0.004 \text{ (megameters)}^7\text{(kiloseconds)}^{-2}$$

The sinusoidal variation of orbital parameters are attributed to the third harmonic component of the earth's field. If the potential associated with this harmonic component is denoted by the following expression:

$$W_3 = \frac{C_{3,0}}{r^4} P_3{}^{(0)} = \frac{C_{3,0}}{r^4}\left(\frac{5}{2} \sin^2\phi' - \frac{3}{2} \sin \phi'\right) \tag{40}$$

where ϕ' is the geocentric latitude, then

$$\sin \phi' = \sin i \sin u$$

where i is the orbital inclination and u is the distance of the satellite from the nodal point. The following expressions are given in Moulton (ref. 12) for r and u in terms of the orbital parameters.

$$r = a(1 - e \cos M + \ldots)$$

$$u = \omega + M + 2e \sin M + \ldots$$

When these substitutions are made, it is found that W_3 has time dependency as the result of terms involving ω and M. The terms involving M represent high-frequency fluctuations and are rejected on the same grounds as the tesseral harmonics were rejected. Thus the following expression for W_3 is obtained:

$$W_3 = \frac{3}{2} \frac{C_{3,0}}{a^4} e \sin i\left(\frac{5}{4} \sin^2 i - 1\right)\sin \omega \tag{41}$$

The time rates of change of the orbital parameters resulting from this perturbational potential have been given in reference 12 (p. 399):

$$\frac{da}{dt} = 0$$

$$\frac{de}{dt} = -\frac{\sqrt{1 - e^2}}{na^2 e} \frac{\partial W_3}{\partial \omega}$$

$$\frac{di}{dt} = \frac{\cos i}{na^2\sqrt{1 - e^2} \sin i} \frac{\partial W_3}{\partial \omega}$$

$$\frac{d\Omega}{dt} = \frac{1}{na^2\sqrt{1 - e^2} \sin i} \frac{\partial W_3}{\partial i}$$

$$\frac{d\omega}{dt} = -\frac{\cos i}{na^2\sqrt{1 - e^2} \sin i} \frac{\partial W_3}{\partial i} + \frac{\sqrt{1 - e^2}}{na^2 e} \frac{\partial W_3}{\partial e}$$

These expressions define the time rates of variations of the orbital parameters resulting from the third harmonic alone. However, the fluctuation in both e and i will induce fluctuations of corresponding periods in $\dot{\Omega}$ and $\dot{\omega}$, through the earth's oblateness terms. Hence, it is necessary to apply corrective terms to the expressions given for $d\Omega/dt$ and $d\omega/dt$. Thus,

$$\frac{d\Omega}{dt} = \frac{1}{na^2\sqrt{1 - e^2} \sin i} \frac{\partial W_3}{\partial i} + \underline{\frac{\partial \dot{\Omega}}{\partial i} \Delta i} + \underline{\frac{\partial \dot{\Omega}}{\partial e} \Delta e}$$

$$\frac{d\omega}{dt} = - \frac{\cos i}{na^2\sqrt{1 - e^2}\sin i}\frac{\partial W_3}{\partial i} + \frac{\sqrt{1 - e^2}}{na^2 e}\frac{\partial W_3}{\partial e} + \underline{\frac{\partial\dot{\omega}}{\partial i}\Delta i} + \underline{\frac{\partial\dot{\omega}}{\partial e}\Delta e}$$

The terms which are underlined represent coupling between the second and third harmonics.

On making the necessary substitutions the following equations are obtained:

$$K_1C_{3,0}\sin\omega = A_e\sin\omega$$

$$K_2C_{3,0}\sin\omega = A_i\sin\omega$$

$$K_3C_{3,0}\cos\omega = A_\Omega\cos\omega$$

$$K_4C_{3,0}\cos\omega = A_\omega\cos\omega$$

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The equation involving de/dt and the equation involving di/dt are, however, interrelated to the extent that angular momentum about the earth's polar axis is preserved. Insofar as the gravitation component is concerned, no change takes place in the semimajor axis and any decrease in inclination must be accompanied by a corresponding reduction in eccentricity. Of these two equations, the one involving eccentricity is retained on the grounds that the measured value of eccentricity is known to a higher degree of precision than is the measured value of orbital inclination. Thus, three equations remain from which to determine the constant $A_{3,0}$. These equations are solved by the method of least squares and yield a value of $A_{3,0}$ of 0.25 ± 0.03 (megameters)^6(kiloseconds)$^{-2}$. Thus the following expression for the earth's gravitational field is obtained when the determined values of the various harmonics are introduced:

$$W = \frac{398.618}{r} - \frac{17.555}{r^3}P_2^{(0)} + \frac{0.25}{r^4}P_3^{(0)} + \frac{1.12}{r^5}P_4^{(0)} \tag{42}$$

In the notation adopted in the section "Theory of External Field"

$$W = \frac{Gm_e}{r_q}\left[\frac{r_q}{r} + J\left(\frac{r_q}{r}\right)^3\left(\frac{1}{3} - \sin^2\phi'\right) + H\left(\frac{r_q}{r}\right)^4 P_3(\sin\phi') + \frac{8}{35}D\left(\frac{r_q}{r}\right)^5 P_4(\sin\phi')\right] \tag{43}$$

where

Gm_e = 398.618 (megameters)^3(kiloseconds)$^{-2}$

$J = 1.6238 \times 10^{-3}$

$H = 2.41 \times 10^{-6}$

$D = 7.5 \times 10^{-6}$

When the earth is assumed to be in a condition of hydrostatic equilibrium, the potential is

$$U = \frac{398.618}{r} = \frac{17.372}{r^3} P_2^{(0)} + \frac{1.95}{r^5} P_4^{(0)} \tag{44}$$

(The potential in question corresponds to a flattening of $\frac{1}{299.8}$.)

The difference therefore defines the departure of the earth from equilibrium. This difference is associated with the deviation potential of

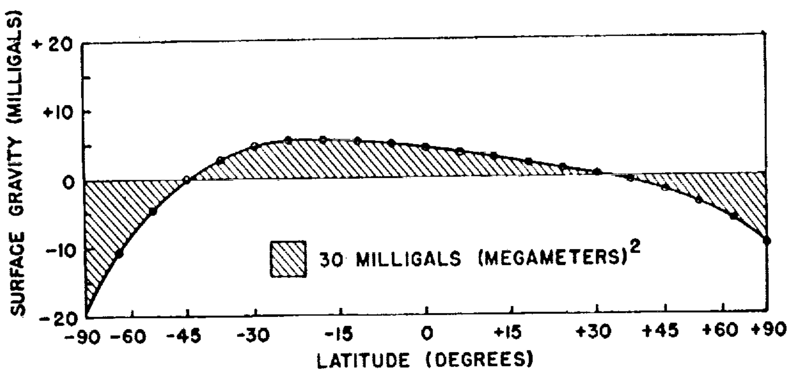
$$V = W - U = -\frac{0.183}{r^3} P_2^{(0)} + \frac{0.25}{r^4} P_3^{(0)} - \frac{0.83}{r^5} P_4^{(0)} \tag{45}$$

By using equations (28) and (31) one can compute the deviation values of surface gravity and the departures of the geoid from the equilibrium spheroid:

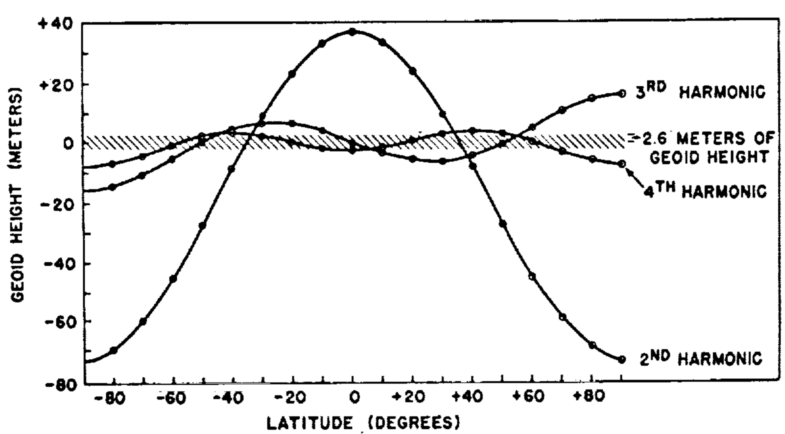
$$N = \frac{1}{9.81}\left[-\frac{0.183}{6.378^3} P_2^{(0)} + \frac{0.25}{6.378^4} P_3^{(0)} - \frac{0.83}{6.378^5} P_4^{(0)}\right] \times 10^6 \text{ meters} \tag{46}$$

$$\Delta g = \left[-\frac{0.183}{6.378^4} P_2^{(0)} + \frac{0.5}{6.378^5} P_3^{(0)} - \frac{2.49}{6.378^6} P_4^{(0)}\right] \times 10^5 \text{ milligals} \tag{47}$$

The results are plotted in sketches 15 and 16 (taken from ref. 11). If the hypothesis of Heiskanen and Vening Meinesz had been valid, then the curves of sketch 15 would have assumed the form of a number of discrete blips both positive and negative not exceeding, however, the area of the shaded square. The curves of sketch 16, on the other hand, would have been constrained to lie within the shaded band except at a number of discrete points where it may on occasion have peaked to 20 meters



Sketch 15



Sketch 16

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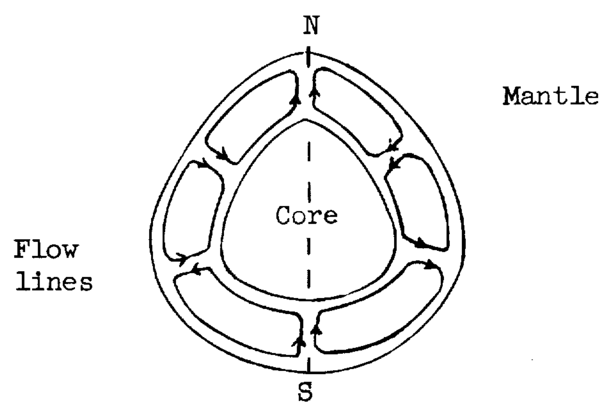
corresponding to the center of large gravity anomalies. The satellite data are clearly in conflict with the Heiskanen hypothesis and support the contention of Jeffreys that the departures from equilibrium of the earth are indeed very appreciable and correspond to wide-scale undulations in the geoid to something of the order of 100 meters.

Geophysical Implications

How then does the earth support such marked departures from equilibrium? Two mechanisms have been proposed: (a) they are supported by convection currents in the earth's mantle, and (b) they are supported by the inherent strength of the material constituting the earth's mantle.

Licht in reference 13 has inquired into the feasibility of mechanism (a). By assuming a system of convection currents similar to that depicted in sketch 17 and assigning a value of 10^{22} poisses for the viscosity of the material comprising the earth's mantle, he has computed the velocity of the convection currents required to support the measured bulges. He finds it to be 3.64 centimeters per year (corresponding to an overturn time of 175×10^6 years). With the velocity of the convection currents and the viscosity known, he then computes the power which is necessary to maintain the convection currents and finds it to be 5.8×10^{18} ergs per second.



Sketch 17

The earth can be pictured as a gigantic heat engine in which these convection currents are maintained by the outflow of heat from the core to the surface. The heat coming from the core is estimated to be 8.2×10^{18} ergs per second.

The Carnot efficiency η is therefore:

$$\frac{\text{Work done}}{\text{Heat extracted from the hot body (earth's core)}}$$

$$\eta = \frac{5.8 \times 10^{18}}{8.2 \times 10^{18}} = 70 \text{ percent}$$

The temperature difference between the core and the surface is also known and from this the ideal Carnot efficiency can be deduced. It was found to be 84 percent. The actual efficiency is much too close to the ideal efficiency to be realistic, and on these grounds the convection hypotheses is rejected as untenable.

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There is no recourse then but to assume that the departures from equilibrium are indeed supported by the inherent strength of the earth's mantle. Had the earth been an elastic body, the stresses within the interior could have been evaluated in a relatively straightforward manner. Thus associated with any specified surface loading of the form $a_n S_n$ where S_n is a surface harmonic of degree n, the greatest stress difference would have been $\lambda_n a_n \sqrt{\overline{S_n^2}}$ where $\overline{S_n^2}$ is the value of S_n^2 averaged over the surface of the sphere and λ_n is approximately equal to unity. For $n = 2$, this maximum stress deviation would have occurred at the core; for $n = 3$, it would have occurred at a distance of 0.4 of the earth's radius; and for the higher harmonics exceeding 3, it would have occurred at a depth of approximately a/n where a is the earth's radius. However, the earth is not an elastic body; the occurrence of thrust and faults in the earth's crust layer testifies to this fact and has led Jeffreys (ref. 3) to adopt the following alternative approach. He has rejected the conditions of elastic deformation and retained only the equations of stress equilibrium. There are an infinite number of stress distributions which will suffice to support a given surface loading. Of this infinite number of stress distributions, one yields the minimum value of the peak stress difference. Such a minimum value defines a lower bound on the stress difference which must occur within the earth's interior. Taking his own numbers for the deviation of the earth from equilibrium which indeed, as has been pointed out earlier, are substantiated qualitatively at any rate by the analysis of orbital data, Jeffrey finds that, if the stress is supported by the entire mantle external to the core, the strength of the material cannot be less than 1.5×10^8 dynes per square centimeter. If, however, it is supported wholly by the mantle up to a depth of 0.1 of the earth's radius, the strength cannot be less than 3.3×10^8 dynes per square centimeter.

MEASUREMENT OF CERTAIN ASTRONOMICAL CONSTANTS

The System of Astronomical Constants

The measurement of the so-called terrestrial constants which serve to define the form and figure of the earth have been discussed in a previous section. These constants are needed in the computation of earth satellite orbits. In this section those additional constants which will be needed in the computation of interplanetary and lunar trajectories and which are involved in problems of space navigation will be discussed. These constants, those of this section and of the previous section, constitute part of the "system of astronomical constants."[7] The use of the word "system" implies an inner structure. This is provided in the present instance by a number of theoretical relationships involving the various astronomical constants. Celestial mechanics differs from most other sciences inasmuch as certain of its theoretical relationships are valid to a higher degree of precision than are the observational values of the parameters which they contain. Such being the case, it is clearly incumbent upon us to make the fullest possible use of these relationships in developing a system of astronomical constants. A logical procedure would be to choose those parameters which can be measured with the highest precision as fundamental parameters and use the pertinent theoretical relationships to derive the others. This appears to be the motivation of DeSitter in reference 15. By virtue of the interlocking of these various constants, any adjustment of the value of any one of them will necessitate adjustments in the others (to preserve theoretical consistency); in short, such an adjustment will permeate the entire system and necessitate complete revision of the astronomical tables. Such a task is not to be lightly undertaken and this explains why revisions to the system of astronomical constants are made so infrequently. The presently accepted values were approved by the International Astronomical Union in 1898.

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There is an additional reason why internal consistency should, at all costs, be preserved even at the expense of accepting certain inaccuracies in the values of certain of the parameters. Thus different investigators may base their calculations on somewhat different parameters. If the parameters they have used are theoretically imcompatible, the results of their investigations will be theoretically incompatible and

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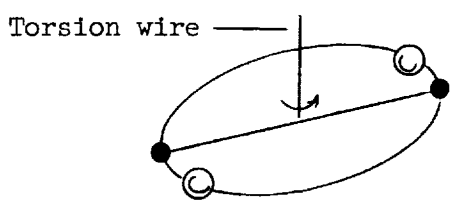
[7]There appears to be no universally accepted definition as to what is embraced by the term "system of astronomical constants." Clemence in reference 14 regards the system of astronomical constants as constituting all parameters of interest in celestial mechanics apart from (a) elements of planetary orbits, and (b) masses of planets other than those of sun, moon, and earth.

the making of a comparison between the separate investigations will be materially more complicated.

The Universal Gravitational Constant (G)

The parameter which is the most fundamental in all matters pertaining to gravitational fields is the universal gravitational constant.

Determination in laboratory units.- A number of determinations have been made of this in the laboratory. The first apparatus for this purpose was built by Rev. John Mitchell in the latter part of the 18th century. It consisted of a wooden arm about 6 feet long bearing at its ends lead balls about 2 inches in diameter. Two larger spheres about 8 inches in diameter were brought up to the proximity of the smaller spheres (see sketch 18) and caused the beam to rotate somewhat against the torsion of the wire. The deflection coupled with a knowledge of the torsional strength of the supporting wire would have enabled the forces between the lead spheres to be evaluated. This in turn would have lead to a determination of G. Rev. Mitchell died before he could make a determination. His apparatus fell into the hands of Cavendish who did make careful experiments in 1797 and 1798.



Sketch 18

The use of such a bulky apparatus for purposes of making such delicate measurements is fraught with uncertainties resulting from distortions of the apparatus, temperature differences, and air currents.

A much more compact apparatus (essentially the same in principle) was designed by Boys in 1895. The arm was only 24 centimeters in length suspended on an extremely fine quartz fiber and bearing at its extremities two gold balls 5 millimeters in diameter. The attracting masses were lead spheres 10 centimeters in diameter.

In more recent determinations, Heyl (1930) and Heyl and Carzanowski (1942), instead of using spheres as the attracting masses, have used cylinders. (Cylinders have the advantage that they can be shaped to a higher degree of precision than can spheres.)

The currently favored value is

$$6.670(1 \pm 0.0007) \times 10^{-8} \frac{\text{cm}^3}{\text{gm-sec}^2}$$

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Determination in terms of astronomical units.- The astronomers' interest is primarily centered on orbital motions relative to the sun. A set of units more suited to the astronomers' particular needs was proposed by Gauss:

Unit of time: Mean solar day.

Unit of mass: Mass of the sun.

Unit of distance: Semimajor axis of the earth's orbit.

The gravitational constant expressed in terms of these units is referred to as the heliocentric gravitational constant and is conventionally denoted by k_s. By applying Kepler's third law to the orbital motion of the earth-moon system,

$$k_s = \frac{2\pi a^{3/2}}{P\sqrt{m_s + (m_e + m_l)}}$$

In terms of the Gaussian astronomical units

$$a = 1$$

$$m_s = 1$$

Uncertainty of k_s arises solely as a result of an uncertainty in P (sidereal period in mean solar days) and in the ratio $\frac{m_e + m_l}{m_s}$. The orbital period is known with extreme precision. The mass ratio $\frac{m_e + m_l}{m_s}$ is much less accurately known; however, it does not sensitively affect the value of k_s. As a result in terms of these Gaussian units, the gravitational constant is known very precisely indeed. More accurate determinations of P and $\frac{m_e + m_l}{m_s}$ have, however, been made since the time of Gauss and rather than adjust k_S which permeates the entire orbital computational procedure, it has been deemed advantageous to incorporate the change simply by revising the unit of length. The revisions necessitate simply introduction of an appropriate scaling factor to all lengths appearing in the end results of the orbital computations.

The current definition of the astronomical unit of length is the mean distance of a fictitious unperturbed planet having exactly the mass and period used by Gauss. It is hardly necessary to point out that the difference between the Gaussian unit and the astronomical unit of length is extremely small. Thus, according to Newcomb, the mean distance of the earth from the sun is 1,000,000,230 astronomical units.

Interrelationship between laboratory units and astronomical units.- By way of summary, the gravitational constant has in effect been measured on the basis of laboratory experiments (in terms of laboratory units) and on the basis of what might be regarded as one of nature's own experiments (in terms of natural astronomical units). The much greater accuracy of the determination in terms of astronomical units testifies to nature's superior skill in setting up an experiment approximating much more closely idealized conditions.

It has always been of interest to tie the laboratory units to the astronomical units; however, of late, it has become a matter of practical concern. Thus, in considering an interplanetary voyage, the initial conditions will be expressed in terms of laboratory units, whereas the terminal conditions (motion of the destination planet) are expressed in astronomical units. Insofar as the conversion of units of mass is concerned, this is not important in trajectory work since it is only the ratio of masses which is significant and not the units in which the individual masses are measured. The laboratory units of time measurement are the same as astronomical units and the question of conversion does not arise here. Only in the conversion of length measurements does an acute problem arise. This is the problem of the determination of the solar parallax to which some consideration will be given later.

Mass of the Earth

The mass of the earth can be determined in one or the other of two ways:

(a) By direct measurement.

(b) Indirectly by measuring accelerations induced on test bodies moving in the gravitational field of the earth and making use of the independently determined value of the gravitational constant.

Thus, it is possible to use either

$$g = \frac{Gm}{r^2} \tag{48}$$

or

$$P = \frac{2\pi a^{3/2}}{G\sqrt{m}} \tag{49}$$

suitably modified to allow for slight perturbational effects. Whether equation (48) or equation (49) is used, the measurement is the same in essence. Thus by using equation (48) accelerations in motion of two bodies along their line of centers are measured and by using equation (49) accelerations associated with orbital motion of one body about the other are measured.

Although the accuracy of the indirect measurements is somewhat superior to that of direct measurement, from the point of historical interest a brief description of two of the more celebrated classical attempts to measure it directly are included.

One of the earliest attempts at a direct determination of the mass of the earth was made by the English astronomer Maskeylyne in 1772. The method consisted of suspending a plumb bob in the proximity of a mountain mass and noting its deflection δ.

$$\delta = \frac{\text{Horizontal pull of the mountain}}{\text{Vertical pull of the earth}} = \frac{Gm_m m/d^2}{Gm_e m/\bar{r}^2} = \frac{m_m}{m_e}\,\frac{\bar{r}^2}{d^2}$$

where

m_m mass of mountain

m_e mass of earth

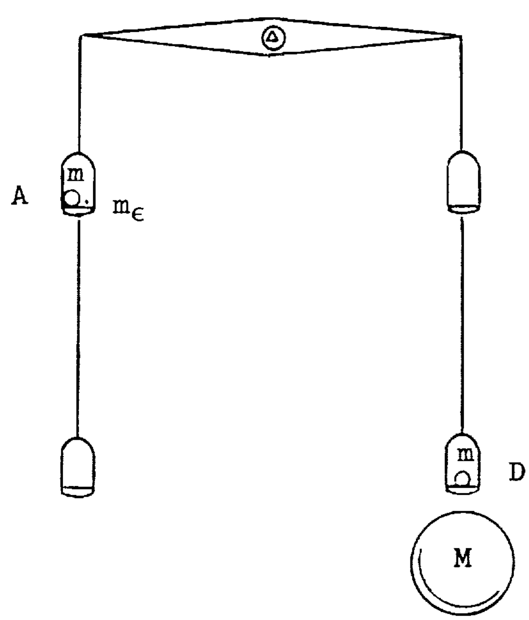
d distance to center of mass of mountain

r radius of earth

By assuming a specific gravity of 2.5, both the mass and the location of its center of mass of the mountain could be estimated and hence the mass of the earth deduced.

The value obtained by Maskeylyne was considerably in error. The error arose principally as a consequence of the condition of isostasy, that is, that mountain masses are at least partially compensated by mass deficiencies under their bases. Indeed, if he had performed his experiment in the proximity of certain mountain ranges he would have found the plumb bob repelled by rather than attracted by the mountain.

A laboratory determination of the mass of the earth was made by Von Jolly in 1881. His apparatus consisted of a balance having two scale pans supported from each arm. A mass m (5 kilograms) was placed in one of the lower pans (that is, D in sketch 19) and an identical mass in the upper pan A supported on the opposite arm. The gravitational pull in the lower of the two masses was somewhat greater than that on the other because of its closer proximity to the earth's center. Balance was reestablished by adding a small weight in pan A. A large mass M (5775.2 kilograms) was then placed beneath pan D. As a result of the gravitational attraction of M on m the balance was again disturbed and was reestablished by placing a very small mass m_ϵ on the upper pan A.



Sketch 19

The force on m_ϵ exerted by the earth must then be equal to the force between m and M. Hence

$$G \frac{Mm}{d^2} = G \frac{m_e m_\epsilon}{\bar{r}^2}$$

from which we can calculate the unknown m_e.

Von Jolly obtained a value of 6.15×10^{27} grams. This value is about 3 percent higher than the currently accepted value of $5.974 \pm 0.005 \times 10^{27}$ grams. The reason for the rather large discrepancy stems from the use of bulky apparatus subject as it is to distortions, unequal temperature, and so forth, to make measurements of extreme delicacy.

Figure of the Moon (Theoretical Considerations)

The form and size of the moon will now be considered. In the case of the earth, the principal deviation from spherical form resulted from the earth's rotation on its axis. In the case of the moon, the rate of rotation is much slower (once in about 27 days) and, in addition, it maintains essentially the same attitude relative to the earth and there is every reason therefore to expect a sizable tidal bulge to exist.

In the theoretical treatment of the earth's figure certain specious reasons were advanced for believing that the earth was essentially in a condition of hydrostatic equilibrium. The same arguments can be invoked in the case of the moon and it is clearly of interest to investigate the equilibrium form which would be assumed by the moon under this hypothesis.

The mathematical problem which poses itself is to determine the form assumed by a fluid mass under the joint action of

(a) Its own gravitational field.

(b) Its rotation.

(c) The earth's attraction.

In the case of the moon, the analysis can be materially simplified by introducing the assumption of uniform density throughout. Such an assumption is justified on the grounds of the smallness of the lunar mass (the internal pressure nowhere attains that existing at the base of the earth's crustal layer). The detailed analysis is presented in reference 3, where it is shown that to a first order the moon is a triaxial ellipsoid with its longest axis pointing toward the earth and its shortest axis coinciding with its axis of rotation. The following expressions are derived for the lengths of the three axes:

longitudinal axis (pointed toward earth),

$$\bar{s}\left(1 + \frac{35}{12}\frac{m_e}{m_l}\frac{\bar{s}^3}{c^3}\right)$$

transverse axis,

$$\bar{s}\left(1 - \frac{10}{12}\frac{m_e}{m_l}\frac{\bar{s}^3}{c^3}\right)$$

L
1
9
7
9

polar axis (axis of rotation),

$$\bar{s}\left(1 - \frac{25}{12}\frac{m_e}{m_l}\frac{\bar{s}^3}{c^3}\right)$$

where

c distance between earth and moon

$\bar{s}$ mean lunar radius

m_e mass of earth

m_l mass of moon

Once its geometrical form is established and the assumption of uniform density is made, the moments of inertia about the principal axes of the lunar ellipsoid can be evaluated. (See ref. 14.)

In particular, by denoting by I_{1l}, I_{2l}, and I_{3l} the moments of inertia about the longitudinal, transverse, and polar axis, respectively, the following relations are obtained:

$$\frac{I_{3l} - I_{1l}}{I_{3l}} = 0.0000375$$

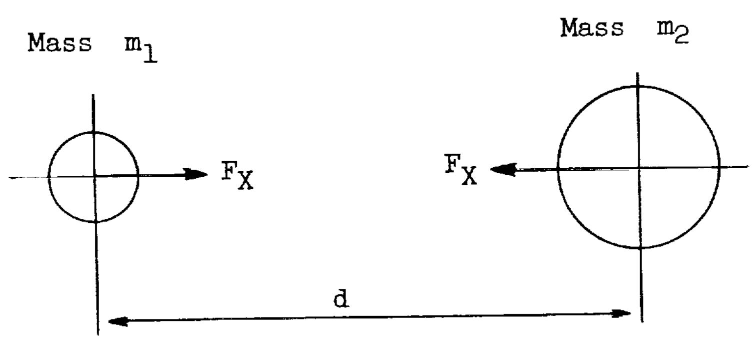
$$\frac{I_{3l} - I_{2l}}{I_{3l}} = 0.0000094$$

$$\frac{I_{2l} - I_{1l}}{I_{3l}} = 0.0000281$$

The principal use of these theoretical investigations is in the determination of the extent to which it tallies with observational data, for in this way the validity of the hypotheses underlying the analysis is checked. In the case of the earth, the observational data were provided by astrogeodetic and gravimetric measurements made on the earth's surface. As yet the capability of making similar measurements on the lunar surface has not been developed. The observational data in the case of the moon is provided by the perturbations in the lunar orbit and the angular motions about its center of mass which result from the moon's lack of sphericity.

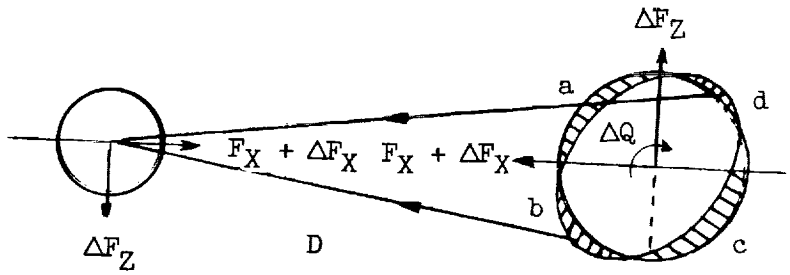
Orbital Perturbations and Librations Resulting From Lack of Sphericity of Gravitating Bodies

Static considerations.- Two spherically symmetric bodies placed with their mass centers at a distance D apart attract one another with a force equal to $G \frac{m_1 m_2}{d^2}$. (See sketch 20.)



Sketch 20

In the event that either or both of the bodies lack spherical symmetry the force of interaction must be amended somewhat. Thus, suppose the mass of body 2 is redistributed to form a spheroid (see sketch 21).

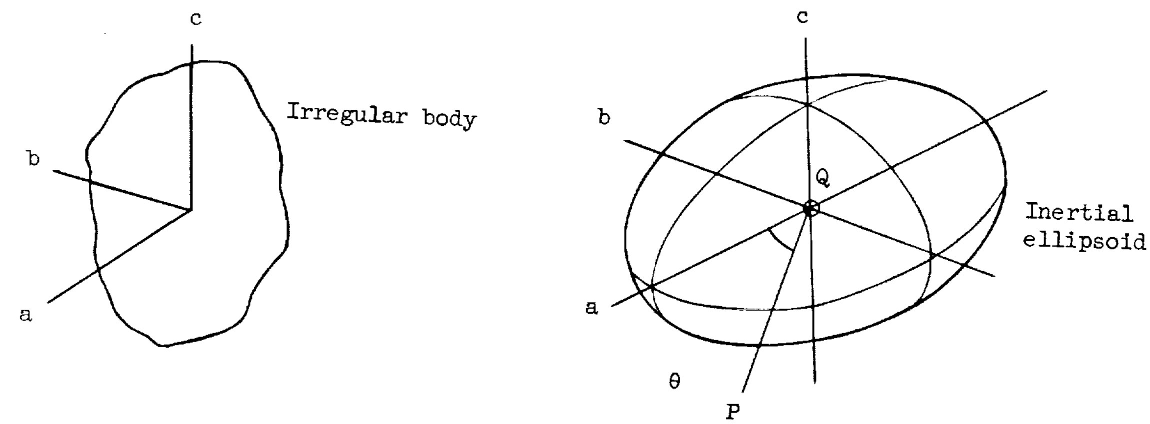


Sketch 21

Specifically suppose the polar mass at a is brought forward to form the bulge at b and the polar mass at c is displaced an equal distance to the rear to form the bulge at d. Displacement of mass from a to b will result in an increase in gravitational pull whereas that resulting from mass transfer to c to d results in a reduction in gravitational pull. By virtue of the fact that the gradient of the gravity field increases as the attracting center is approached, these two effects are not quite equal but result in a slight net increase in overall force ΔF_X. In addition, since the force on b exceeds that on d there results a net force in the upward direction ΔF_Z and a net torque ΔQ about the mass center of the oblate body in the direction indicated.

The resultant force on the spherically symmetric body acts through the center of the mass. By applying the laws of statics it is noted that the component of the resulting force along the line of centers is equal to $F_X + \Delta F_X$ and the component in the vertical direction is ΔF_Z downwards. Moreover, since the net moment on the entire system is zero, it follows that $\Delta Q = d \cdot \Delta F_Z$.

Analytic expressions for these force and moment increments can be obtained by appealing to MacCullagh's formula which serves to define the potential of a body of arbitrary shape at a "distant point." (See sketch 22.)



Sketch 22

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$$\psi = G\left(\frac{m^2}{d} + \frac{I_X + I_Y + I_Z - 3I}{2d^3}\right) \tag{50}$$

where

I	moment of inertia about arbitrary axis QP
I_X	moment of inertia about principal axis Q_a
I_Y	moment of inertia about principal axis Q_b
I_Z	moment of inertia about principal axis Q_c

Since in the case presently under consideration the body is spheroidal $I_X = I_Y$ and $I = I_X \sin^2\theta + I_X + I_Z \cos^2\theta$. Hence

$$I_X + I_Y + I_Z - 3I = 2I_X + I_Z - 3\left(I_X \sin^2\theta + I_Z \cos^2\theta\right)$$

$$= -3\left(I_Z - I_X\right)\left(\cos^2\theta - \frac{1}{3}\right)$$

Thus

$$\psi = G\left[\frac{m_2}{d} - \frac{3\left(I_Z - I_X\right)}{2d^3}\left(\cos^2\theta - \frac{1}{3}\right)\right] \tag{51}$$

If this expression for the potential of a spheroidal body is compared with the more complete expression given in section "Theory of External Field" what is meant by the phrase at a large distance appearing in the enunciation of MacCullagh's theorem can be seen. It is a distance sufficiently great to ensure that contributions arising from third and higher harmonics are insignificant. If, instead of a unit mass at P (as is implied in the definition of potential) there is a mass m_1; the mutual potential energy is actually given by

$$\psi = G\left[\frac{m_1 m_2}{d} - \frac{3}{2}\,\frac{m_1\left(I_Z - I_X\right)}{d^3}\left(\cos^2\theta - \frac{1}{3}\right)\right] \tag{52}$$

Force and moment components are determined by the following derivatives:

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$$F_X + \Delta F_X = -\frac{\partial\psi}{\partial d} = G\left[\frac{m_1 m_2}{d^2} - \frac{9}{2}\,m_1\,\frac{I_Z - I_X}{d^4}\left(\cos^2\theta - \frac{1}{3}\right)\right]$$

$$\Delta F_Z = -\frac{1}{d}\,\frac{\partial\psi}{\partial\theta} = G\left(-\frac{3}{2}\,m_1\,\frac{I_Z - I_X}{d^4}\,\sin^2\theta\right)$$

$$\Delta Q = -\frac{\partial\psi}{\partial\theta} = G\left(-\frac{3}{2}\,m_1\,\frac{I_Z - I_X}{d^3}\,\sin^2\theta\right)$$

Note that $\Delta Q = (d)\left(\Delta F_Z\right)$ as is demanded by conditions of statical equilibrium.

Dynamic considerations.- First of all consider the orbital motion of a spherically symmetric body about an oblate body. In this case the forces which are operative on the spherical satellite are modified slightly as a consequence of the oblateness of the attracting mass and hence the orbit is perturbed somewhat from Keplerian form. However, the force acting on the satellite at all times acts through the center of mass and no angular motions are induced. Consider next the complementary situation in which a spheroidal satellite is in orbit about a spherically symmetric mass. In this case the oblateness of the satellite induces changes in the force components resulting in perturbations in the orbital motion. It also gives rise to a torque about the center of mass producing angular motions (librations) about the center of mass. It is of interest to compare the magnitude of these angular motions with the magnitude of the perturbations in orbital motion. A measure of the amplitude of the librations is provided by the ratio $\Delta Q/Ma^2$ where ΔQ is the applied torque, M is the mass of the satellite, and a is a representative body radius. In considering the perturbations in orbital motion it suffices to consider the influence of the force increment ΔZ. This condition gives rise to a torque $(R)(\Delta Z) = \Delta Q$ about the attracting center and operative on the orbit as a whole. A measure of the perturbation to which the orbit is subject as a consequence is provided by the ratio $\Delta Q/MR^2$, where R is the orbital radius. Hence, $\dfrac{\text{Libration}}{\text{Perturbation in orbital motion}}$ is of the order of $\left(R^2/\bar{r}^2\right)$. In the case of the earth-sun system this ratio is of order of magnitude $\left(\dfrac{90,000,000}{4,000}\right)^2$, that is, 5.05×10^8; and thus, although the rocking of the equatorial plane associated with the earth's oblateness is very significant, changes in the orbital motion are virtually insensible.

With regard to the earth-moon system, because of the moon's closer proximity, the ratio is of a smaller order of magnitude. Thus, $\dfrac{\text{Librations resulting from mass oblateness}}{\text{Orbital perturbations resulting from mass oblateness}}$ are of the order of $\left(\dfrac{250,000}{1,000}\right)^2$ that is, 62,500. Although the librations are much larger in this instance, the orbital perturbations due to moon's oblateness are detectable and are indeed utilized, as will be seen later, to provide a quantitative measure of the deviation of the moon from spherical form.

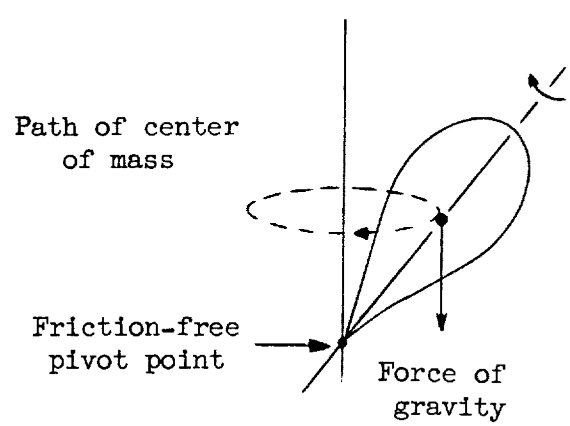
By way of summary then, if the two cases which have been discussed separately are superimposed and an oblate body orbiting about an oblate attracting mass is considered, both the oblateness of the satellite and of the attracting mass contribute to the orbital perturbations to which the satellite is subject. However, insofar as the librations about the center of mass is concerned, only the oblateness of the satellite contributes here.

Motion of earth.- The line of intersection of the earth's equatorial plane and the plane of the ecliptic serves as a basic datum line in celestial mechanics. It pierces the celestial sphere in the equinoctial points. If these planes maintained the same orientation in space, the year of the seasons (from solstice to solstice) would be exactly equal to the year determined by the heliacal rising and setting of the stars (times when certain constellations rise and set with the sun).

Hipparchus around 125 B.C. on examining records embracing some several hundred years noted that the solsticial (year of seasons) was shorter than the heliacal year (sidereal year). From this he deduced a westward migration of the equinoctial points of 36 seconds of arc per year. The data he used were somewhat rough and his estimate was some 30 percent low. Such a migration of the equinoxes points to either a rotation of the equatorial plane or the plane of the ecliptic or both.

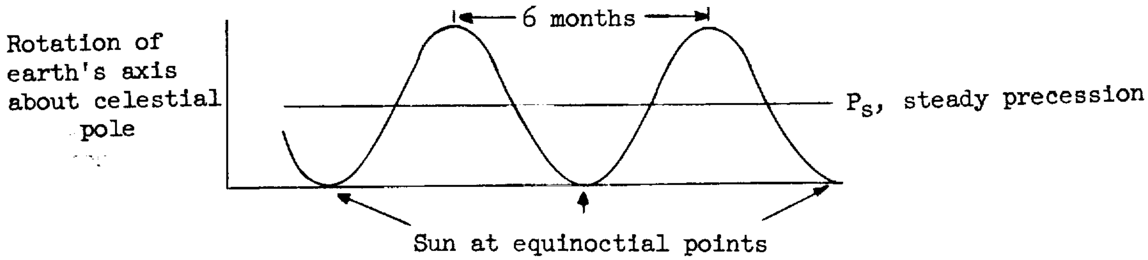
Later observations showed that celestial latitudes of the stars were subject only to very slight variation and the effect was principally the result of a rotation of the equatorial plane. Newton was the first to advance a qualitative explanation. Euler subsequently provided a rigorous and elegant mathematical demonstration of it.

The reason for this precession motion is most clearly exhibited in terms of the analogy with a spinning top. Suppose that a top is spinning on an essentially friction-free pivot. (See sketch 23.) The gravitational torque instead of causing the top to topple induces a steady precession of the axis of rotation about the vertical. (In addition, there is a superimposed oscillatory movement of the axis, the so-called nutation. This is in the nature of a transient and can be disregarded in the astronomical problem since such transients have had ample time to subside.) Reverting to the earth-sun system, the earth acts as the spinning top. The sun's attractive force on the equatorial bulge gives rise to a torque tending to swing the bulge into the plane of the ecliptic. As a result, the earth's axis of rotation precesses about the vertical. The situation in this case is somewhat more complex than our top analogy to the extent that the torque is not constant; thus, when the sun is coincident with one or other of the equinoctial points, the torque falls to zero. The net outcome is that the movement of the axis assumes a sinusoidal form as depicted in sketch 24 consisting of a steady precession P_s and a superimposed sinusoidal fluctuation of a 6-month period. The moon itself has a similar effect and also produces a steady precession P_l and a superimposed oscillatory motion of a biweekly



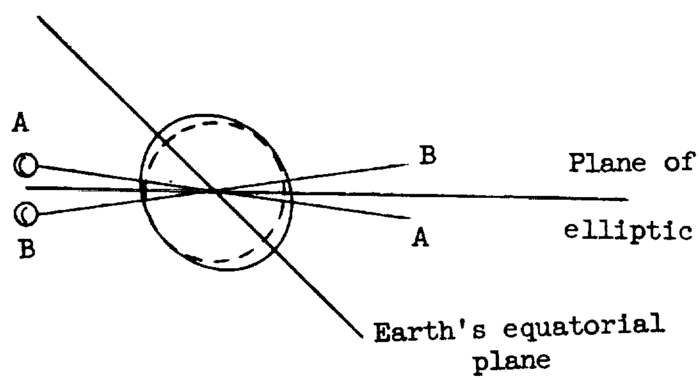
Sketch 23

period. By combining these results, the earth's axis is subject to a steady motion $P_s + P_l$ referred to as the lunisolar precession[8] with certain fluctuating components which are referred to as nutations.[9]



Sketch 24

There is an additional effect resulting from the rotation of the moon's orbital plane over a period of 18.6 years. Note the situation along the line of intersection of the equatorial plane and the ecliptic. (See sketch 25.) At a certain epoch the moon's orbital plane is at AA and the lunar effect is minimal. Some 9.3 years later the orbital plane is at BB and the effect is maximized. Thus there is an additional periodicity in the motion of the earth's axis of 18.6 years. This effect constitutes the dominant nutational effect.



Sketch 25

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[8]The lunisolar precession amounts to 50".37 per year, the moon's contribution exceeding by roughly a factor of 2 that of the sun.

[9]The use of the word nutation in this context differs from that in discussion of spinning tops inasmuch that they here refer to forced motions and not to transients.

To complicate matters further the plane of the ecliptic is subject to slight motion. This motion arises as a consequence of the other planets tending to pull the earth's orbital plane into the planes of their respective orbital motion. The effect is very complex and gives rise to secular changes rather than periodic ones. At the present time it is producing an eastward migration of the equinoctial points of about $0''.11$ per year; this rate is itself subject to slow variation. This migration is referred to as the planetary precession. The general precession is the result of superimposing lunisolar precession and planetary precession and amounts to $50''.26$ per year. There is an additional contributing factor; the solar system is itself rotating about the center of the galaxy and completes a revolution in about 170×10^6 years. In other words, the background of stars viewed relative to a strictly inertial frame is rotating about the galactic pole at a rate of $0''.74$ per 100 years. To obtain the motion of the equinoctial points relative to an inertial system, due consideration must be given to this component of motion and the general precession suitably amended.

Not only is the earth's orbital plane subject to secular rotation about the vector defining the angular momentum of the entire solar system (which serves to define the normal to the invariable plane) but there are secular changes of its inclination,[10] the line of apsides,[11] and the eccentricity. A rough qualitative plot of the latter which serves to typify the irregular character of secular changes is presented in sketch 26. It is of interest to note that, although the secular changes are erratic, there are theoretical bounds between which the values of the orbital parameters are constrained to lie. Thus in the case of the earth's orbit, its eccentricity must fall in the range of 0 to 0.0677, and the inclination of the orbital plane relative to the invariable plane can never exceed $3^\circ6'0''$. (See ref. 17.)

No mention has been made to the secular variation of the semimajor axis. Laplace believed he had satisfactorily demonstrated that this particular parameter was not subject to change and hence rigorously establishes the stability of the solar system. Admittedly, Laplace's analysis disregards all dissipative terms. The solar tides will promote changes in the earth's angular momentum about its axis which must be compensated by a corresponding change in the angular momentum associated with the earth's orbital motion. However, the angular momentum associated

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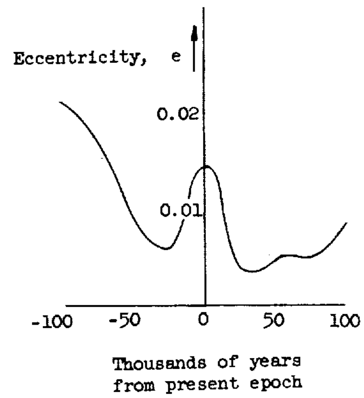
[10]The inclination of the plane of the ecliptic to the invariable plane is presently about $1^\circ35'$. It will diminish over the next 20,000 years to a minimum value of about $47'$. Thereafter it will begin to increase.

[11]Line of apsides is currently revolving eastward at a rate which, if it were to remain constant (which it will not by reason of the secular character of the change), would carry it around in 108,000 years.



Sketch 26

with the earth's orbital motion is so overwhelmingly greater than the angular momentum associated with the earth's rotation on its axes that any changes arising in the orbital motion resulting from solar tidal friction are quite undetectable. There exists a much more cogent reason for doubting the validity of Laplace's analysis. This reason stems from the fact that the series development he was using was not convergent as he thought but divergent of a kind that is known as asymptotic. At the moment the question remains unresolved. Since the effect is theoretically unpredictable and observationally undetectable, there is no point in pursuing the matter further.

<u>The motion of the moon</u>.- The motion of the moon is, to a good accuracy, summarized in Cassini's three empirical laws (see ref. 18):

1. The moon rotates uniformly about an axis which is fixed with respect to the moon itself. The period of this rotation is identical with the sidereal period of the moon in its orbit, namely 27.321661 days.

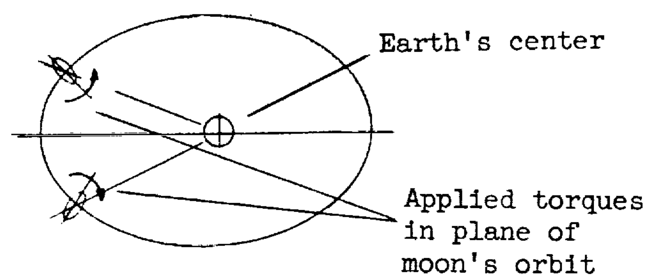
2. The pole of the lunar rotation z makes a constant angle ($1^{\circ}35'$) with the pole of the ecliptic Z which may here be regarded as a fixed point on the celestial sphere.

3. In consequence of the nearly uniform regression of the lunar node on the plane of the ecliptic and the nearly constant inclination of the lunar orbit ($5^{\circ}9'$) the pole of the moon's orbit P is known to describe a small circle about Z in a period of 18.6 years. The arc of the great circle zP contains also the pole Z. In other words, the planes of the lunar orbit and the lunar equator intersect on the ecliptic, the latter plane being intermediate between the two former.

This last-mentioned fact elicited some surprise when it was first enunciated by Cassini. D'Alembert sought an explanation without avail.

It engaged the attention of other mathematical giants Legendre, Laplace, and Poisson and through their joint efforts an explanation was finally given. The precession of the lunar axis about the vertical is the direct outcome of the lack of sphericity of the moon and indeed it can be used to make a fairly definitive determination of $\frac{I_Z - I_X}{I_Z}$ (namely, 0.00062 to within an accuracy of 1 percent) in precisely the same way as the precession of the earth's axis leads to a definitive determination of $\frac{I_Z - I_X}{I_Z}$.

There exists another angular motion about the center of mass which holds some interest. By virtue of the difference of the moments of inertia about the radial and transverse axes in the plane of the lunar orbit coupled with the circumstance that the moon's orbit about the earth is somewhat eccentric and thus gives rise to nonuniform angular velocity, a pendulum-type oscillation is induced in the plane of the orbit. (See sketch 27.) Both the earth and the sun contribute to this rocking motion and thus give rise to two terms, one having a monthly periodicity and the other having an annual periodicity. The former falls below the threshold of observational detection and, although the annual term (moon's annual libration in longitude) can be detected, it amounts to only 1 mile displacement of the center of the lunar disk (corresponding to about 1" of arc at the earth's distance). As a result of its smallness and the difficulty of its accurate determination by virtue of the boundary irregularity of the lunar disk, it yields a relatively poor determination of $\frac{I_Y - I_X}{I_Z}$, the uncertainty being a sizable fraction of the whole amount.



Sketch 27

The principal perturbations in the orbital motion itself, resulting directly from the moon's oblateness, are three in number:

(1) Motion of lunar node.

(2) Motion of lunar perigee.

(3) Monthly variation in the moon latitude.

The third effect is inextricably entangled with the obliquity of the ecliptic. This effect can be seen by supposing the moon to be moving in the earth's equatorial plane; its latitude would be subject to a monthly variation by virtue of the obliquity of the ecliptic. Unfortunately, the obliquity of the ecliptic (nor the time variations to which it is subject) is not known with sufficient precision to permit an accurate determination of the monthly variation in the moon's latitude. On these grounds Jeffreys (ref. 3) suggests that it should not be employed in arriving at estimates of the moon's dynamical ellipticities.

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Consider next the contribution of the oblateness of the earth and moon to the motion of the lunar node and perigee. The most dominant contribution to the rotation of node and perigee is to be attributed to the sun (69,672".04 per year and 146,426".92 per year, respectively).

An estimate of the magnitude of the contribution resulting from the moon's oblateness will now be made. Elsewhere, it is shown that librational motion of the lunar axis about the vertical should be some 62,500 times the orbital perturbation. The librational motion of the polar axis is such as to result in a complete circuit in 18.6 years (that is, 69,672".04 per year). On this basis an orbital perturbation of the order of seconds of arc per year may be expected and this is indeed what is observed. These numbers serve to illustrate the difficulty of the determination of the lunar dynamical ellipticities involving the sorting out a contribution which amounts to only one part in several tens of thousandths of the total and it testifies to the extreme precision of astronomical measurements that analysis of these perturbations provides useful supplementary information on the dynamical ellipticities of the moon.

The best available values of the dynamical ellipticities to date are

$$\frac{I_Z - I_X}{I_Z} = 0.0006269 \pm 0.0000027$$

$$\frac{I_Y - I_X}{I_Z} = 0.000118 \pm 0.000057$$

If uniformity of density is assumed, these values imply that the lunar axis pointing toward the earth is about 1/4 mile longer than the transverse equatorial axis and this in turn exceeds the polar axis by about 1 mile ($\approx$1.6093 km).

Comparison of Theoretical and Observational Values of the Ellipticities of Moon

Comparing the observational values given above with those given in section "Figure of the Moon" which have been derived theoretically discloses a considerable discrepancy. The assumption underlying the theoretical treatment which is most suspect is that relating to hydrostatic equilibrium. If the difference is attributed to this condition, the internal stresses associated with such marked departures from hydrostatic condition will demand material of at least the strength of brick to support them. This in turn leads us to question the validity of assuming hydrostatic equilibrium in the case of the earth. On this point there is independent evidence from the analysis of Vanguard I orbital data and other artificial satellites that the earth does indeed exhibit marked departures from hydrostatic equilibrium.

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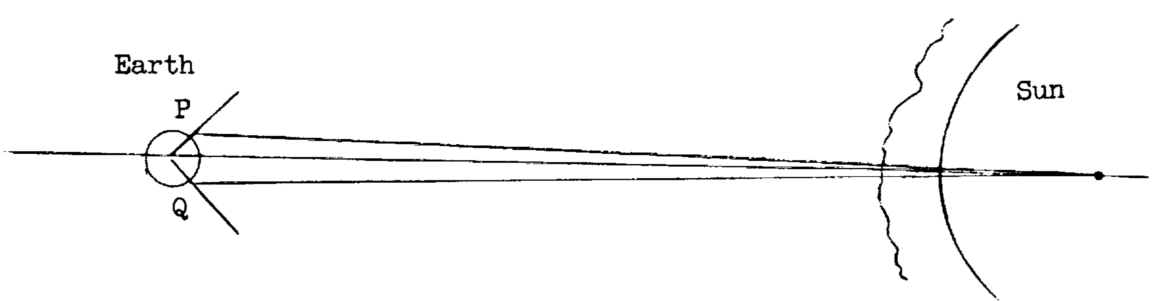
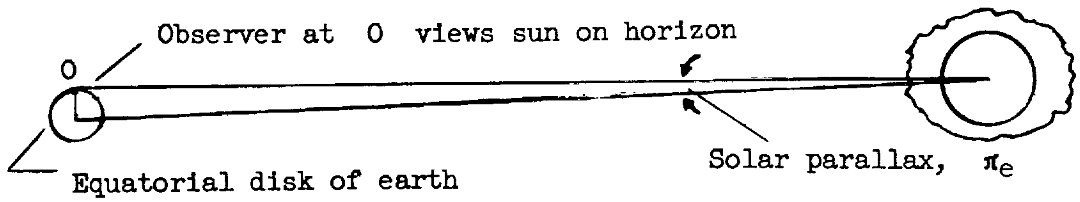
Solar Parallax

The need to determine the earth's distance from the sun in terms of laboratory units has already been stressed. One way in which this can be done in principle would be to make simultaneous observations of the center of the sun's disk from two distant points P and Q on the earth's surface. (See sketch 28.) A simple trigonometric calculation would then give the distance to the sun in terms of earth radii and since the earth's radius is known in laboratory units to a high precision this is tantamount to a measure of the distance in laboratory units. In other words this



Sketch 28

calculation would provide a direct measure of the sun's mean equatorial horizontal parallax[12] (solar parallax for short) defined in sketch 29.



Sketch 29

The solar parallax (π_e) is so small (about 8 seconds of arc) that this trigonometric procedure would not lead to a very accurate determination. The accuracy can, of course, be materially improved by triangulations on a body which is closer to the earth and the distance of which is known accurately in astronomical units. Such a favorable opportunity[13] was provided by the asteroid Eros in 1931 when it came within 13,000,000 miles of the earth. Use of Eros had the additional merit that Eros is so small that to all intents and purposes it was a point source of light; thus, the uncertainty in determining the center of an illuminated disk of finite dimensions was eliminated. Based on observations made on Eros during this close encounter, Spencer-Jones obtained a value of the solar parallax of $8''.790 \pm 0''.001$.

Rabe in 1950 made a dynamical determination of the solar parallax. As a preliminary to describing the method used by Rabe, it will be necessary to derive a certain theoretical relationship.

If Kepler's third law is applied to the motion of the earth-moon system about the sun,

$$4\pi^2R^3 = G(m_s + m_e + m_l)P_e^2 \tag{53}$$

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[12]This parallax was mean in the sense that the earth is at its mean distance from the sun, equatorial in the sense that the observation is made from a point on the equator, and horizontal in the sense that the sun is viewed when located on the horizon.

[13]Even more favorable opportunities will present themselves in the next 10 years. Thus: 1566 Icarus within 4,000,000 miles in 1968; 1620 Geographos within 4,000,000 miles in 1969.

where

R	mean orbital distance
P_e	sidereal year
G	universal gravitational constant
m_s	mass of sun
m_e	mass of earth
m_l	mass of moon

If Kepler's third law is applied to orbital motion of moon about the earth,

$$4\pi^2 l^3 = G(m_e + m_l)P_l^{\,2} \tag{54}$$

where

l	mean lunar distance
P_l	sidereal month

Equations (53) and (54) may be rewritten in the form

$$G(m_s + m_e + m_l) = 4\pi^2 \frac{R^3}{P_e^{\,2}}$$

$$G(m_e + m_l) = 4\pi^2 \frac{l^3}{P_l^{\,2}}$$

Hence,

$$Gm_s = 4\pi^2 \frac{R^3}{P_e^{\,2}}\left(1 - \frac{l^3}{R^3}\frac{P_e^{\,2}}{P_l^{\,2}}\right) \tag{55}$$

However,

$$g_q = \frac{Gm_e}{r_q^{\,2}}\left(1 + \epsilon - \frac{3}{2}\kappa\right)$$

or

$$Gm_e = \frac{g_q r_q^{\,2}}{1 + \epsilon - \frac{3}{2}\kappa} \tag{56}$$

Dividing equation (55) by equation (56) yields

$$\frac{m_s}{m_e} = \frac{4\pi^2 R^3}{g_q r_q^{\,2} P_e^{\,2}}\left(1 + \epsilon - \frac{3}{2}\kappa\right)\left(1 - \frac{l^3}{R^3}\frac{P_e^{\,2}}{P_l^{\,2}}\right)$$

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Assume that the solar parallax in seconds is $\tilde{\omega}$

$$\frac{r_q}{R} = \tilde{\omega} \sin 1'' \quad \text{or} \quad R = \frac{r_q}{\tilde{\omega} \sin 1''}$$

Therefore,

$$\frac{m_s}{m_e + m_l}\tilde{\omega}^3 = \frac{4\pi^2 r_q}{(\sin 1'')^3 P_e^{\,2}} \; \frac{1 + \epsilon - \frac{3}{2}\kappa}{1 + \mu}\left(1 - \frac{l^3}{R^3}\frac{P_e^{\,2}}{P_l^{\,2}}\right)$$

$$= 2.26444 \times 10^8 \tag{57}$$

to a high accuracy where μ is the ratio of the moon's mass to earth's mass.

In the light of this relationship it is seen that a determination of the ratio $\frac{m_s}{m_e + m_l}$ leads directly to a determination of comparable accuracy of the solar parallax and, of course, vice versa. This relationship provides the basis of a dynamical determination of the solar parallax. Rabe analyzed the perturbations in the orbit of Eros over the interval 1926 to 1945 which embraces its close encounter with the earth-moon system. From his analysis he was able to deduce a value of the mass ratio $\frac{m_s}{m_e + m_l}$ and this in turn led to a value of the solar parallax of 8".798 ± 0".0004. It is to be noted that the difference between the values obtained by Spencer-Jones trigonometrically and that obtained dynamically by Rabe exceeds the probable error associated with either determination. The disparity remains unexplained.[14] To compound the confusion Newcomb, toward the close of the last century, calculated the value of $\frac{m_s}{m_e + m_l}$ from the perturbation of the orbit of Venus, specifically the rotation of its orbital node, resulting from the attraction of the earth-moon system. This procedure led to a value of solar parallax of $8\overset{''}{.}759 \pm 0\overset{''}{.}010$ which is in disagreement with all other determinations. Discovery of the source of this anomaly in the motion of Venus remains an outstanding challenge to those working in the field of dynamical astronomy.

There are, of course, a number of other techniques for determining the solar parallax. One is based on an examination of the parallactic inequality of the lunar motion. This condition arises as a consequence of the sun being at a finite distance (hence, the name parallactic inequality). Thus the solar perturbation of the lunar motion is more pronounced when the moon is in conjunction than when it is in opposition. Such an effect clearly depends upon the relative distances of the moon from the earth and the earth from the sun and can be used for determining the latter.

Of these various determinations the one which is currently favored is that of Rabe. There is every reason to expect, however, that within the next few years more precise determinations of the solar parallax will be made. One technique which offers promise is the use of an artificial probe fitted with a transponder. Once the probe's orbit has been definitively established in terms of astronomical units, a radar pulse is beamed to it, on receipt of which it immediately responds by emitting a signal back to earth. The total elapsed time multiplied by the velocity of light divided by 2 will provide an accurate measure of the distance in laboratory units.

One source of uncertainty which may be raised by the reader is that space is not a perfect vacuum but is filled with charged particles which may serve to modify the velocity of an electromagnetic wave. Although the point is valid in principle, it is not a problem of practical concern. Thus Herrick (ref. 19) has estimated the error resulting from this cause to be less than the uncertainty of the velocity of light in a vacuum.

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[14]The trigonometric value is, of course, influenced by the value assigned to the earth's oblateness. Within recent months, on the basis of analyses of orbital perturbations of artificial satellites, there has been a major revision in the oblateness factor. Instead of helping to reconcile the two measurements, it results in an even greater disparity. (See ref. 20.)

Since this paper was written, more accurate determinations of the solar parallax has been made by making radar measurements of the distance of Venus. These more recent and more accurate determinations are in good agreement with Rabe's value.

A ground-based measurement of potentially considerable accuracy has been proposed by Brouwer and Lilley. (See ref. 21.) It consists simply of comparing Doppler shifts of the 21-centimeter hydrogen line emitted by a specific interstellar cloud at 6-month intervals. This would lead to a very accurate determination of the earth's orbital velocity. (See also ref. 19.)

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Lunar Parallax

The moon's mean equatorial horizontal parallax (lunar parallax) can be obtained trigonometrically or dynamically. The best trigonometric determination is given in reference 2 as

$$\sin \pi_{\ell} = 0.016592278\left(1 + \frac{3}{2} f' + p'\right) \tag{58}$$

where f' represents the uncertainty in the earth's oblateness (specifically the deviation of the true value from the adopted value[15] of 1/297) and p' is the inherent inaccuracy associated with the observational technique which is estimated to be $\pm 18 \times 10^{-6}$.

The dynamic determination is based on the use of the formulas

$$4\pi^2 \ell^3 = G(m_e + m_{\ell})P_e^2$$

and

$$g_q = \frac{Gm_e}{r_q^2}\left(1 + \epsilon - \frac{3}{2}\kappa\right)$$

Dividing one by the other yields the relation

$$\sin^3 \pi_{\ell} = \frac{r_q^3}{\ell^3} = \frac{4\pi^2 r_q}{g_q P_e^2} \frac{1 + \epsilon - \frac{3}{2}\kappa}{1 + \mu}$$

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[15]The currently favored value of the earth's oblateness obtained from analysis of motions of artificial satellites is 1/298.3 with an uncertainty which is extremely small.

and leads to the expression given in reference 2

$$\sin \pi_{l} = 0.016592715\left(1 + \frac{1}{3} a' - \frac{1}{3} g' + \frac{2}{3} f' - \frac{1}{3} \mu'\right) \qquad (59)$$

where

a' deviation of equatorial radius from adopted value of 6,378,270 meters (represented by a factor $\pm 10 \times 10^{-6}$)

g' deviation of equatorial gravity from adopted value of 978,036.8 milligals (represented by a factor $\pm 3 \times 10^{-6}$)

f' deviation of earth's oblateness from adopted value of 1/297 (represented by a factor $\pm 4 \times 10^{-6}$)

μ' deviation of lunar-earth mass ratio (suitably modified to embrace perturbational effects) from adopted value of 1/81.375 (represented by a factor $\pm 4 \times 10^{-6}$)

By adopting these uncertainties as given in reference 2, it is seen that dynamical determination is a more definitive one than is the trigonometric determination.

Ratio of Lunar Mass to Earth's Mass

The precession and nutation[16] of the earth's axis are both dependent on the earth's oblateness and the lunar mass; thus, the equations for the determination of these two unknowns are provided.

A more accurate estimate of the lunar mass, however, can be obtained in the following way. The earth and moon rotate about their joint center of mass which is displaced some 5,000 kilometers from the earth's center. By virtue of this rotational motion the sun when observed from the earth appears to undergo a slight oscillatory motion with a monthly period. This motion is referred to as the "lunar inequality in the sun's longitude." The amplitude is small (about 6".45), and does not permit a very precise measurement. By sighting on an object closer to the earth, the to-and-fro motion is amplified and a more accurate determination can be made. Use of the observational data obtained on Eros during its close encounter of 1931 yielded a more accurate evaluation of the lunar inequality in sun's longitude.

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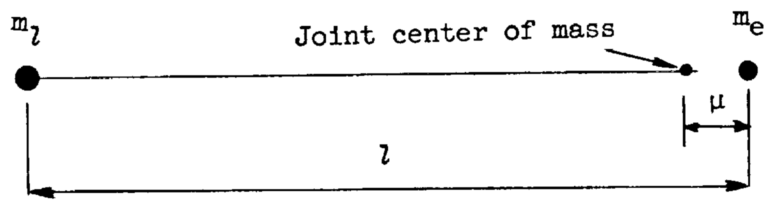
[16]Reference is made here to the principal nutational component of period 18.6 years.



Sketch 30

Note in sketch 30 that

$$(m_e + m_l)\mu = m_l l$$

$$\mu = \frac{m_l}{m_e + m_l} l$$

Lunar inequality L in the sun's longitude is thus

$$\begin{aligned} L &= \frac{\mu}{R} \\ &= \frac{m_l}{m_e + m_l} \frac{l}{R} \\ &= \frac{m_l}{m_e + m_l} \frac{r_q/R}{r_q/l} \\ &= \frac{\mu}{1 + \mu} \frac{\pi_s}{\pi_l} \end{aligned}$$

Depending on whether the value for the solar parallax of Spencer-Jones or of Rabe is used, values of μ of 8.127 ± 0.025 or 81.375 ± 0.026 are obtained. Since Rabe's value is believed to be the more accurate, it is the latter value which is currently favored.

RELATIVISTIC CONSIDERATIONS

There are two reasons for introducing relativity into a discussion of gravitational fields:

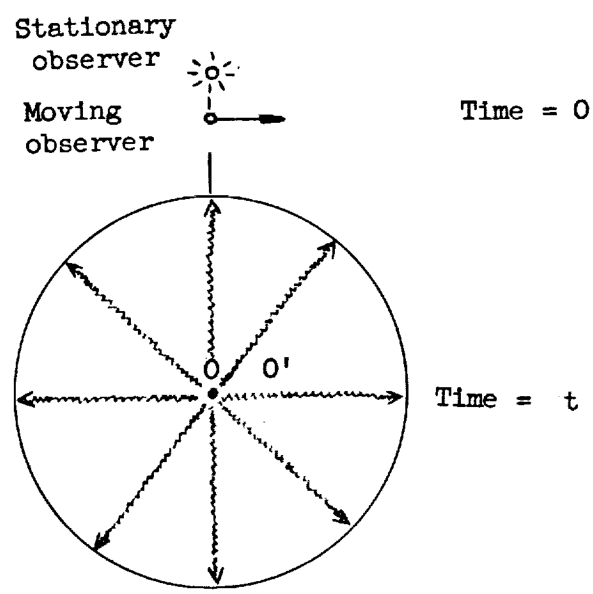
(a) It provides a more logically consistent picture of the nature of gravitation.

(b) Artificial satellites can be used to provide checks on both the special and general theories of relativity.

In the pedestrian approach adopted in the present text our interest lies primarily in the second reason. However, it was thought to be desirable to give brief résumés of both the special and general theories of relativity. In this regard considerable liberties have been taken
with the theory, if by so doing, the concepts involved were rendered more L
readily assimilable. One such liberty is in our interpretation of invar- 1
iance of interval $x^2 + y^2 + z^2 - c^2t^2$ in different coordinate systems. 9
The assumption is made that this invariance implies that the coordinate 7
systems are orthogonally related (as indeed they would be if the terms 9
were all positive). Actually, they are related by a hyperbolic Minkowski transformation.

Special Theory of Relativity (Enunciated by Einstein in 1905)

Consider two observers O and O'. (See sketch 31.) O' is moving past O with uniform speed V. At the instant of passing O initiates a light pulse. After the elapse of time t the light pulse will have advanced a distance ct relative to O. What is O's estimate of the distance? Classical theory would indicate it to be $(c - v)t$, experiment shows it to be ct.



Sketch 31

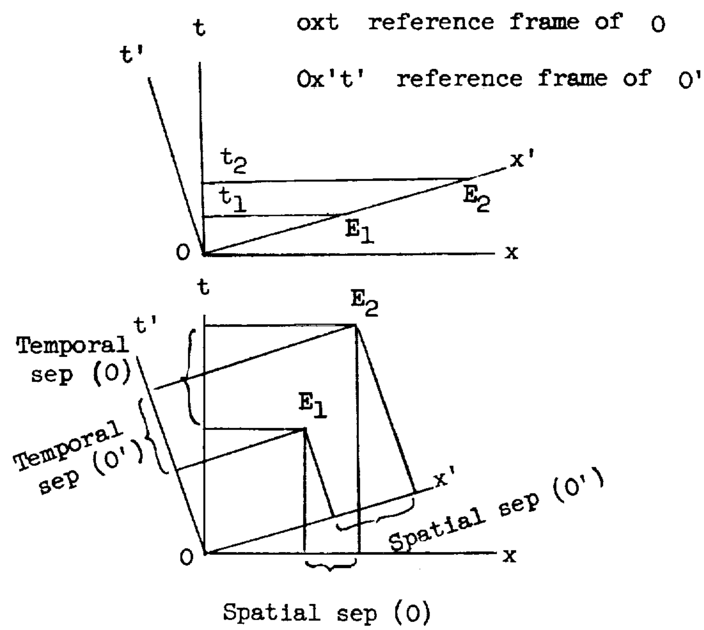
To put the matter a little more formally, after the elapse of time t the light pulse is spread over a closed surface. The equation of the surface relative to O is $x^2 + y^2 + z^2 = c^2t^2$ and relative to O' is $x'^2 + y'^2 + z'^2 = c^2t'^2$; that is,

$$x^2 + y^2 + z^2 - c^2t^2 = x'^2 + y'^2 + z'^2 - c^2t'^2 \tag{60}$$

It is known, however, that if in two dimensions

$$x^2 + y^2 = x'^2 + y'^2 \tag{61}$$

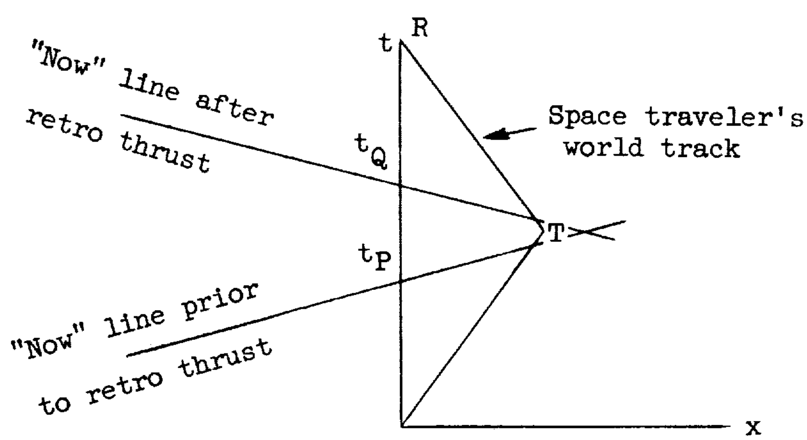
then xy and $x'y'$ define two orthogonal reference frames rotated through some angle with respect to one another. If the liberty of disregarding the sign differences in equations (60) and (61) is permitted, then equation (60) tells us that the reference frames of O and O' differ only to the extent that they are rotated relative to one another. To simplify the discussion as far as possible, only one space dimension and one time dimension will be considered. The reference frames of the two observers assume the form shown in sketch 32.



Sketch 32

To observer O' events E_1 and E_2 both occur at $t = 0$, that is, "Now." Observer O, on the other hand, informs us that these events happened, respectively, at times t_1 and t_2. Simultaneity is thus a fictitious concept contrary to our intuitive notions. This is the converse concept to that introduced by Newton; to wit, it is impossible to detect a state of uniform motion, that is, the same place at two different times. The latter tells us, "It is impossible to determine the same time at two different places." It is curious that one idea is intuitively acceptable and the other is not.

In sketch 32, note that the spatial and temporal separations of the events are assigned different measures by each of our two observers. It can be expressed equivalently by stating that measuring scales appear to contract and clocks slow down when they are set into motion. This brings us to the so-called twin paradox. It may be described briefly as follows. Consider two twins, one climbs into his rocket ship, blasts off, and goes hurtling off to some distant part of the universe. There he applies his retrothrust, reverses his direction, and comes hurtling back. The question arises as to whether his twin brother who stayed behind has aged relative to himself. There are some who regard such a belief as almost heretical. They adopt the same standpoint as did the inquisitors with regard to the teachings of Copernicus. They contend that the knowledge of biology is not sufficient to discuss the relative aging of living organisms. This difficulty can be circumvented by using identical clocks instead of identical twins. But it might be argued that when the clocks are in uniform motion each will appear to the other to be running slow. This is true so long as our observers are in relative motion. However, the two observers are not wholly equivalent. It is the traveller who experiences acceleration at blast off, at the terminal point, and at the completion of his journey and not his brother. Glancing at sketch 33 oxt is the reference frame of our stay at home observer OTR is the space voyager's path. Bear in mind that in uniform motion our "Now" lines



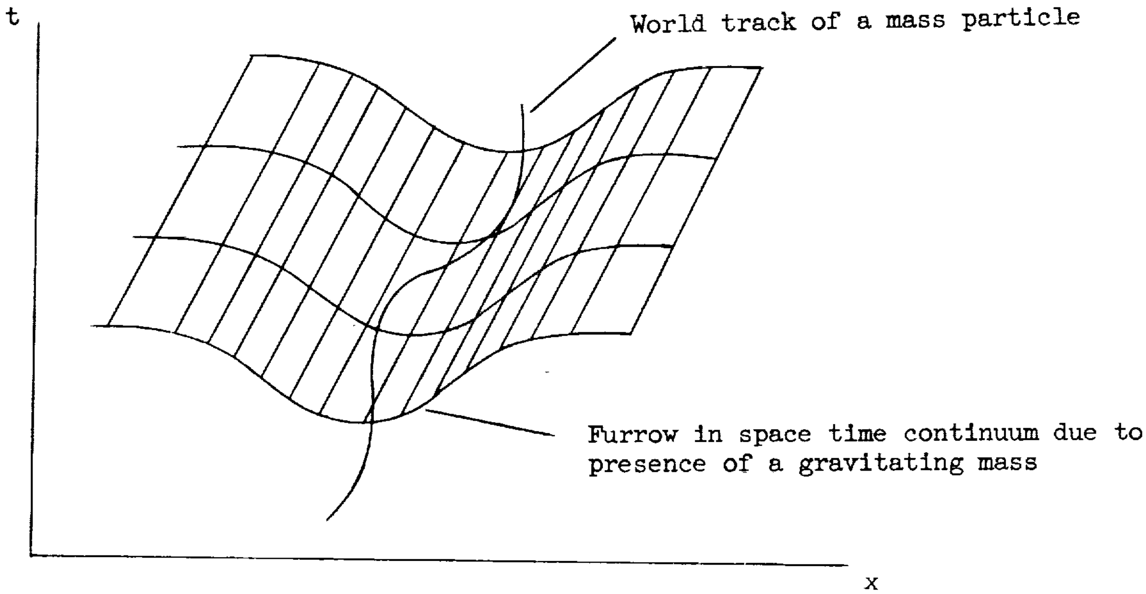
Explanation of twin paradox

Sketch 33

are slewed around. Thus, just prior to reaching T our space traveler's "Now" line runs from T to P, and it will be his belief that the earth clock is reading t_P. The instant after firing his retrorocket his velocity has changed and his "Now" line runs from T to Q and the earth clock seemingly is reading t_Q. Thus in the instant of application of retrorocket the earth clock has advanced from t_P to t_Q. This latter adjustment more than compensates for the slow running of the earth clock during the outward and return journeys and accounts for the relative aging of his twin brother during his absence.

Outline of General Theory of Relativity

The special theory achieved a spectacular unification in both the fields of dynamics and electrodynamics. It did, however, muddy the waters in certain respects, particularly with reference to the law of gravitation $F = \frac{GM_1M_2}{r^2}$. This law is not invariant under the Lorentz transformation which describes the transition from one inertial frame to another. Thus gravitational phenomena will differ as one passes from one inertial frame to another and these differences could be utilized to distinguish one state of uniform motion from another. (Not only Einstein but Newton himself would have objected to this.) Moreover, what meaning can be assigned to r, that is, by whose measuring scale is it to be measured? Einstein circumvented this impasse in a very ingenious way. The following analogy was given by Eddington (ref. 22). Suppose that the belief in a flat earth had persisted down to the present day. A Mercator's chart would then have been regarded as giving an accurate description of the disposition of objects on the earth's surface. If the map were centered on the United States of America, then no serious disparities would have arisen with reference to local journeys, that is, journeys within the confines of the United States. On the other hand, travellers to distant parts of the earth would have noticed that they were able to cover apparently great distances with little expenditure of effort. The scientists would doubtless have explained the phenomenon as the result of the action of a mysterious force operative in distant parts of the earth. The situation with regard to gravity is very similar and just as our fictional force owes its existence to our attempting to force a spherical surface into a flat surface, so gravity owes its existence to our forcing what is in reality a curved space time into a flat space time. The presence of matter puts an indentation or rather a furrow in space time extending along its world track. (See sketch 34.) If a particle were projected along the furrow with a velocity which is not too large, it will clearly oscillate from one side to another in the manner depicted. Since the observers are moving in the time dimension, the particle will appear to describe closed ellipses corresponding to the empirical law introduced by Newton. The mind of the "practical" man is apt to rebel at the concept



Sketch 34

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of curved space time. However, it is to be remembered that Gauss, one of the greatest and yet one of the most practical minded of mathematicians of all time, was not convinced of the flatness of space. Indeed, he proposed an experiment whereby the issue might be decided. He proposed to light three bonfires on the tops of three distant mountains, triangulate, and determine whether the sum of the three angles equalled 180°. The experiment was never carried out nor would it have yielded any measurable deviation if it had been. You might say it is one thing to admit the possibility of curved space time and quite another to demonstrate its actuality. What proof have we that space time is indeed curved? There have been three deductions of general theory which have been subject to experimental confirmation:

(a) Precession of the perihelion of Mercury.

(b) Deflection of light rays passing close to the sun.

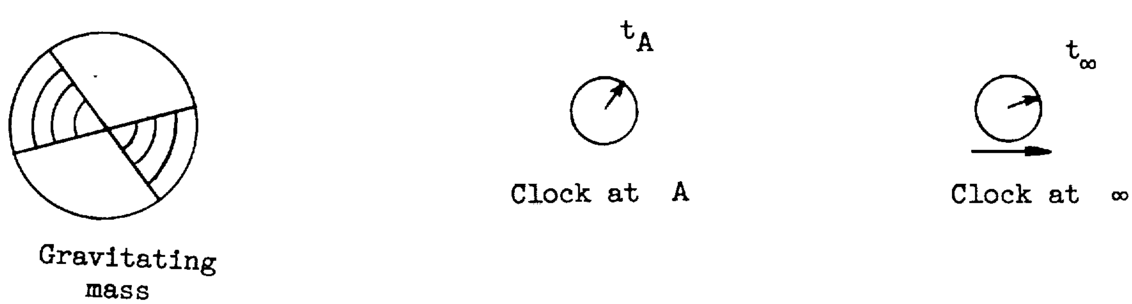
(c) Red shift in solar spectral lines.

With regard to (a) the orbit of Mercury was subject to an unaccountable rotation amounting to 43 seconds of arc per 100 years (the rotation occurring in the direction of the satellite's motion). General relativity explained this discrepancy almost exactly, and thus provided a very excellent quantitative check on the theory. The check provided by (b) is somewhat less definitive. The consensus appears to be that a statistical average of measurements agrees with Einstein's prediction to within ±5 percent. The check (c) is of even more dubious nature and is more qualitative rather than quantitative. Admitting the possibility that the agreement in relation to the single check (a) could be fortuitous, it is clearly highly desirable that further checks be provided.

Checks on Theory of Relativity

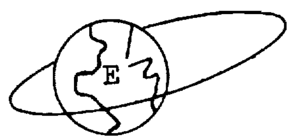
How are further checks to be made? The curvature of space can be detected in the same way as that for a surface. Thus, draw a circle and measure both its circumference and its radius and form the ratio; if the space be curved, the ratio will deviate somewhat from 2π. Einstein's formulas being accepted at their face value, if the circumference of the sun and its radius are measured, then the ratio would deviate from 2π in about the sixth place of decimals. If the same experiment were performed with reference to the earth, the disparity would appear in about the tenth decimal place. To put it another way, if the earth is assumed to be truly spherical in form and its circumference were measured and its radius deduced by dividing by 2π, the result would differ from the true radius (as determined by direct measurement) by about 5 millimeters. Thus, insofar as spatial distortion is concerned, the effect is very slight.

Not only can the distortion of space time be detected by using measuring scales but also by using clocks. Thus, if (see sketch 35) a gravitating mass exists at O and if a clock is placed at infinity, the clock at A relative to the clock at infinity appears to be running slow. The relation expressing the relative rates is $\frac{t_A}{t_\infty} = 1 + U_A$.



Sketch 35

The latter term is the potential energy associated with point A relative to infinity (hence it is negative) nondimensionalized by dividing through by c^2. To be a little more specific, consider the earth as the gravitating mass and a satellite orbiting about it. (See sketch 36.)



Clock at $\vec{\infty}$

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Sketch 36

Relative to the observer at infinity the clock in the satellite is running slow -

(a) By virtue of its being in a gravitating field

(b) By virtue of its motion

$$\frac{dt_s}{dt_\infty} = 1 + U_s - T_s \tag{62}$$

A clock on the earth's surface also appears to be running slow relative to the observer at infinity. In this instance the effects of motion resulting from the earth's rotation are quite negligible. Hence

$$\frac{dt_e}{dt_\infty} = 1 + U_e \tag{63}$$

Bearing in mind that U and T are extremely small and dividing equation (63) by equation (62) an expression relating the reading of the satellite's clock to that of the clock located on earth's surface is obtained.

$$\frac{dt_s}{dt_e} = 1 - U_e + U_s - T_s \tag{64}$$

Relative rates of time keeping are a function of orbit parameters and location of the satellite within its orbit.

$$\frac{dt_s}{dt_e} = F(a, e, \theta) \tag{65}$$

If equation (64) is integrated over a complete revolution, the result is found to be solely a function of semimajor axis a.

$$\frac{\text{Periodic time as estimated by satellite's clock}}{\text{Periodic time as estimated by earthbound clock}} = 1 + \frac{Gm_e}{c^2\bar{r}} - \frac{3}{2}\frac{Gm_e}{c^2 a} \tag{66}$$

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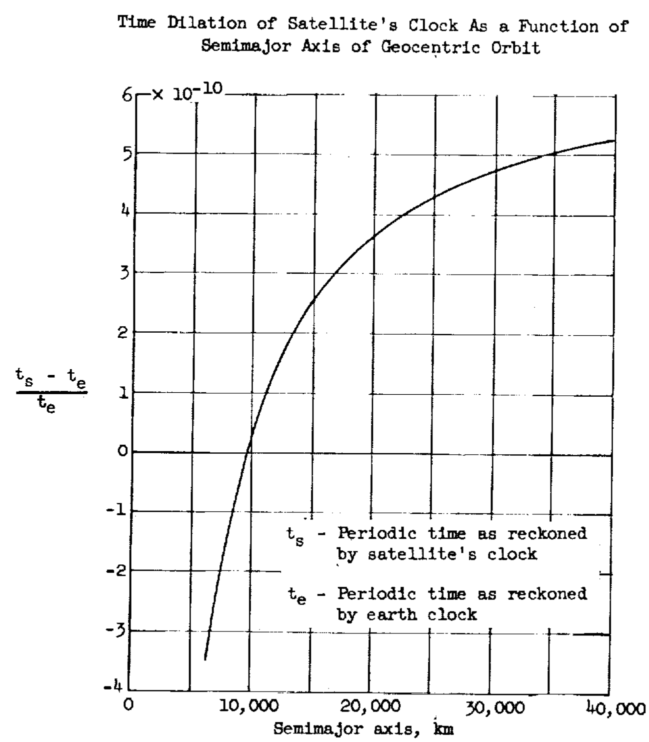
The results are plotted in sketch 37.



Sketch 37

Note that for very tight orbits (orbits hugging close to the earth's surface) the satellite's clock appears to an earth observer to be running slow. For large orbits the satellite clock appears to be running fast. The crossover point corresponds to a semimajor axis of 9,540 kilometers (that is, in the case of a circular orbit this point corresponds to a height above the earth's surface of 3,170 kilometers or approximately 2,000 miles). Note also the magnitude of the disparity being of the order of one part in 10^{10}.

Whenever atomic clocks are developed to the extent that it is possible to keep constant tally on their timekeeping as they pursue their paths in the orbit (rather than rely on cumulative effects over a large number of complete orbits) then a sinusoidal variation will be detected, the amplitude of the variation being a function of eccentricity. In sketch 38 the variation in timekeeping from point to point of the satellite's orbit has been plotted. The orbit in question has a semimajor axis of 20,000 kilometers and an eccentricity of 0.6. In this instance in the neighborhood of perigee it is running slow and at apogee it is running fast. On the average (defined by the horizontal line), it is running fast. Note that the average rate does not correspond to the arithmetic mean but lies appreciably closer to the time rate at apogee. This is the result of the satellite moving slower at apogee and hence spending most of its time in this neighborhood.

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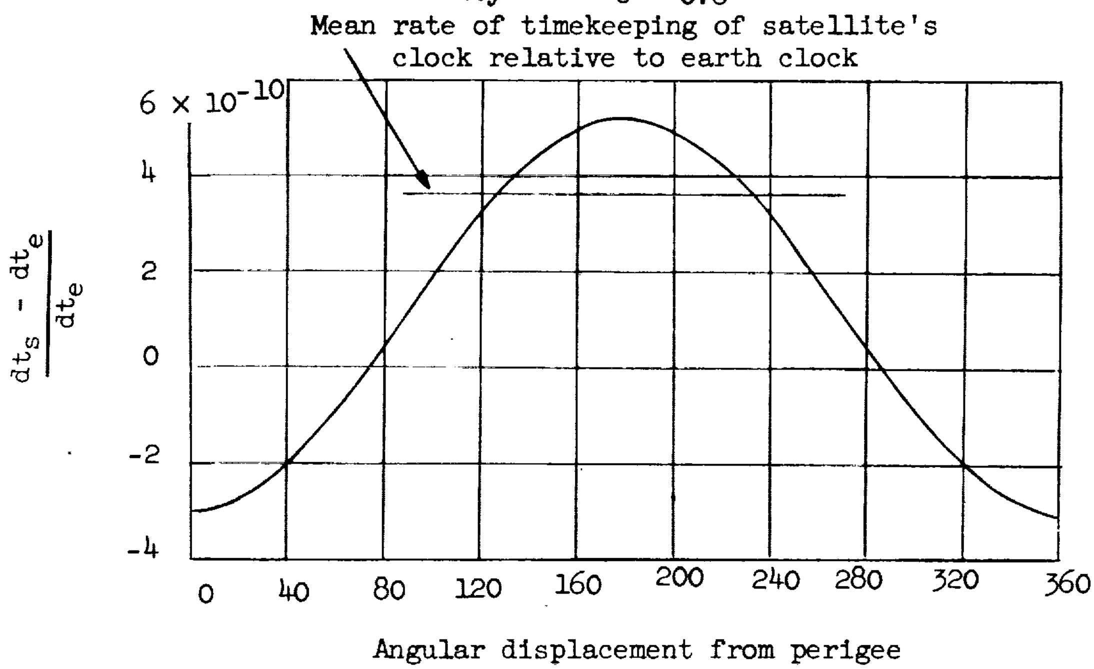
Relative rates of timekeeping as a function of position in orbit

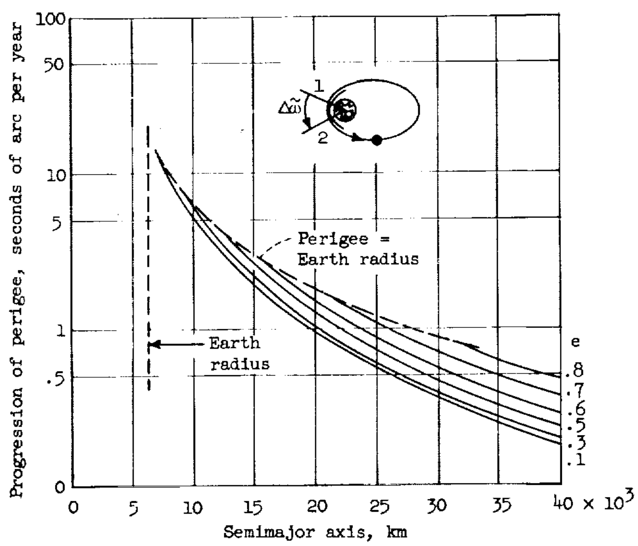
Semimajor axis $a = 20{,}000$ km
Eccentricity $e = 0.6$



Sketch 38

In addition to purely kinematical tests of the kind just described, other checks can be provided. One such is associated with the advance of the perigee point (similar to the effect detected in orbit of Mercury). In the neighborhood of spherically symmetric gravitating bodies the orbit does not quite close on itself. (See sketch 39.) The perigee advances by a small amount $\delta\omega$ during each revolution in the direction of the satellite's motion as given by the formula

Relativistic precession of perigee point of earth satellites



Sketch 39

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$$\Delta\tilde{\omega} = \frac{6\pi G m_e}{c^2 a(1 - e^2)} \qquad (67)$$

where

G	universal gravitational constant
m_e	mass of earth
c	velocity of light
a	semimajor axis of orbit
e	eccentricity of orbit

The advance in the perigee point expressed in seconds of arc per year is plotted against semimajor axis for various values of orbit eccentricities. By virtue of the smallness of the orbit the effect can be made as large as 15 to 20 seconds of arc per year (compared with 43 seconds advance per 100 years as displayed by Mercury). The difficulties associated with carrying out such an experiment should not be underrated. Thus, as a result of the oblateness of the earth and deviations from homogeneity in its internal structure, the orbit is appreciably perturbed:

(a) There is a rotation of the plane containing the orbit, and

(b) There is an advance of the perigee point within the plane of its rotation.

(Both of these effects may amount to several degrees per day.) It is the latter effect which is inextricably entangled with the relativistic effect and, since it can amount to several hundred degrees per year, then to detect the 20 seconds of arc, the potential associated with earth will need to be known to extreme precision. (It may well be that continual changes in its internal structure may suffice to mask the effect completely.) A further check proposed by Ginzburg (ref. 23) is associated with the relativistic perturbation to the satellite's orbit resulting from the earth's rotation. To gain some insight into the nature of this effect, let us consider an electromagnetic analogy. Prior to Maxwell's derivation of electromagnetic field equations, electrical phenomena were assumed to obey the universal Coulombic law $F = c \dfrac{e_1 e_2}{r^2}$ and electrical disturbances were assumed to be propagated with infinite speed; the similarity with gravitation is striking. Thus, prior to Einstein's derivation of gravitational field equations, gravitational phenomena were assumed to obey the universal Newtonian law $F = c \dfrac{m_1 m_2}{r^2}$ and gravitational disturbances were assumed to be propagated with infinite speed. It is generally agreed that gravitational effects in common with electromagnetic effects are propagated with the speed of light.

Consider now a charged particle moving in the proximity of a charged sphere. If the charged sphere is set into rotation, it will set up a magnetic field which will interact with that of the charged particle and its motion will as a consequence be modified. If relativity is to be believed, a similar effect will manifest itself in the case of a particle moving in the neighborhood of a gravitating body; thus, even though the body be uniform and spherical, its rotation will modify to a slight extent the motion of the particle in question. In this instance, the regression of the perigee point is given by the formula

$$\Delta\omega = - \frac{8}{5} \frac{\sigma \bar{r}^2 G m_e}{c^2 a^{3/2} (1 - e^2)^{3/2}} \qquad (68)$$

where

σ rotational speed of earth

$\bar{r}$ earth's mean radius

A plot of $\Delta\omega$ as a function of a and e is given in sketch 40. Note that the most favorable choice of orbit parameters yields only an effect of order of 0.1 second of arc per year. (It is interesting to note that in the case of Mercury the effect only amounts to 0.01 second of arc per 100 years.)

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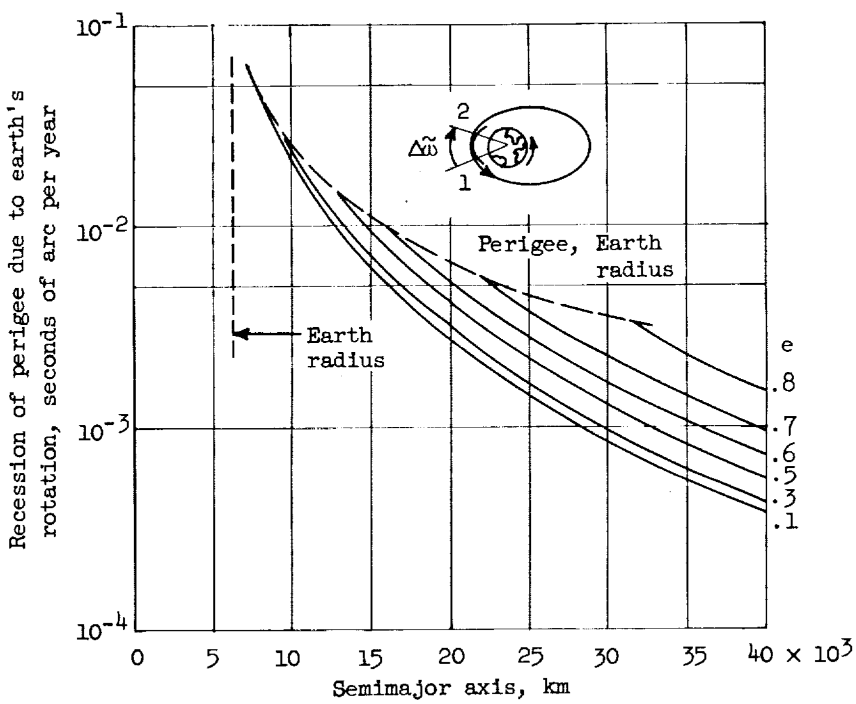
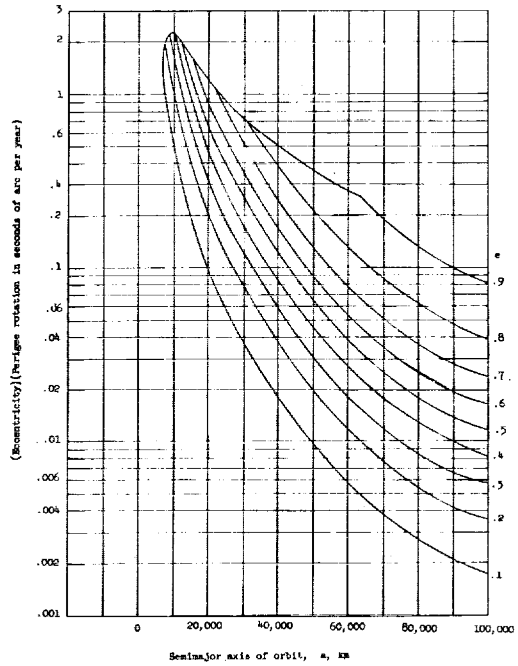
Relativistic recession of perigee point due to earth's rotation



Sketch 40

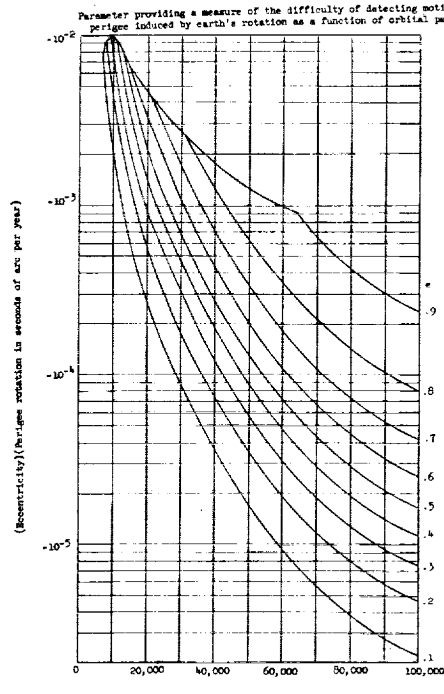
Sketches 41 and 42 differ from sketches 39 and 40 only to the extent that the ordinate defines the product of eccentricity and perigee movement. It is this parameter which provides a reasonably accurate measure of the ease of detection of the relativistic effects in question. Thus it is to be noted that the orbit which most sensitively exhibits these relativistic effects and which therefore recommends itself for their detection and measurement has a semimajor axis of 10,000 kilometers and an eccentricity 0.35.

Parameter providing a measure of the difficulty of detecting relativistic advance of perigee of earth satellite as a function of orbital parameters.



Sketch 41

Parameter providing a measure of the difficulty of detecting motion of perigee induced by earth's rotation as a function of orbital parameters.



Sketch 42

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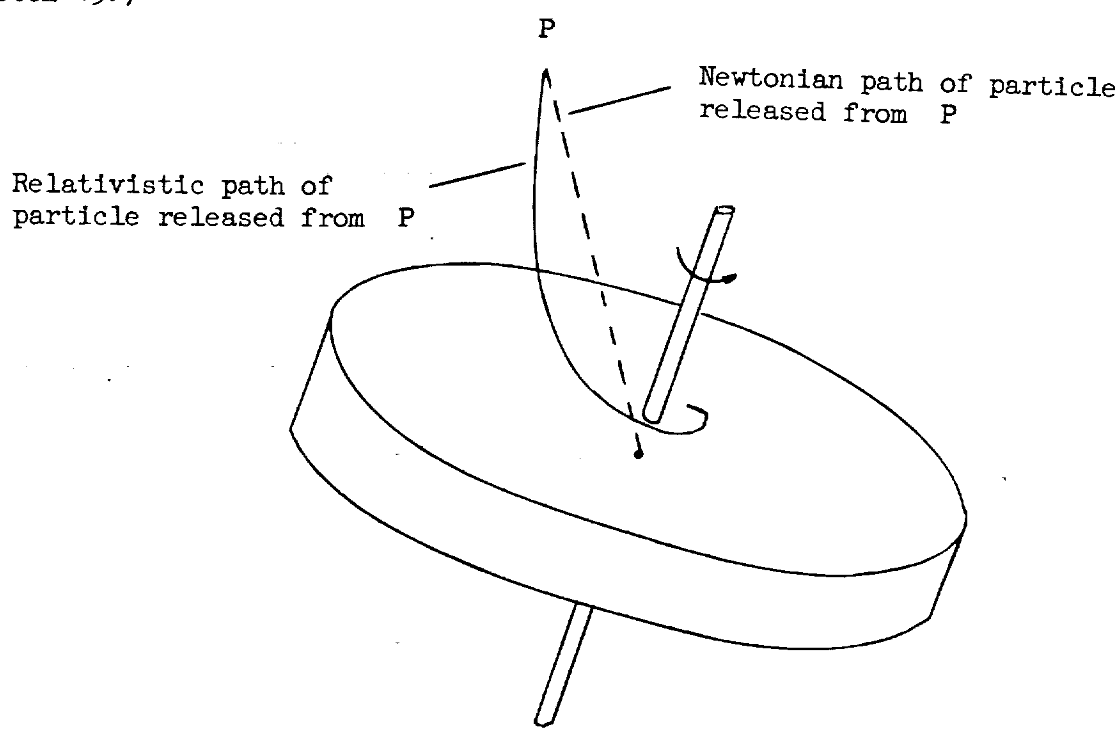
In addition to these there are other experiments which could be made. Thus, if a rapidly rotating flywheel is placed aboard a space ship and a particle is placed in its proximity both are in a condition of free fall. However, they will slowly come together under their mutual gravitational attraction. Relativity tells us, however, that instead of proceeding more or less directly toward the center of mass of the flywheel the particle will tend to be dragged around by the flywheel. (See sketch 43.)



Sketch 43

Current Status of the Theory of Relativity

There is a popular belief that insofar as its dynamical aspects are concerned the theory is complete and all that is needed is experimental verification of the basic laws. Nothing could be further from the truth. Two dynamical problems are currently giving rise to considerable theoretical research.

(1) The two-body and many-body problem.

(2) Gravitational waves.

Two-body problem.- The status of the two-body problem in relativity theory is similar to that of the three-body problem in classical mechanics. Seemingly, the source of the trouble is the following. In his classic paper of 1915, Einstein gave his law of gravitation $G_{\mu\nu} = 0$ $\left(\text{replacing Newton's law } F = \dfrac{Gm_1m_2}{r^2}\right)$ and his law of motion of mass particles $\dfrac{d^2x_\alpha}{ds^2} + \left\{\mu\nu,\alpha\right\}\dfrac{dx_\mu}{ds}\dfrac{dx_\nu}{ds} = 0$ (replacing Newton's law $F = ma$). This equation has to be satisfied at the location of the particle. In the mathematical equation the particle is regarded as a mass point (that is, an infinite singularity) and the terms appearing in the equation are not even defined. This situation is one which arises in classical field theory. By virtue, however, of the linearity of the field equations it is permissible to disregard the self-field of the particle and the derivatives can be taken as those associated with the field which would exist if the particle were to be removed. As a result of the nonlinearity of the gravitational field equations, this expedient is no longer open to us. In a paper published in 1937, Einstein, Infeld, and Hoffmann devised an iteration technique which could be applied to the two-body and many-body problems. As so often happens in applying iteration procedures to the solution of complex sets of nonlinear equations, it is difficult to know what the end result is. Thus, by way of example, the earliest solution contained so-called radiation terms. These were subsequently found to be fictitious, that is, they could be transformed away by switching to another reference.

Gravitational waves.- One thing seems plain if gravitational waves do exist, they are going to be troublesome to detect. Compare the detection of electromagnetic waves with those of gravity waves. In the case of the electromagnetic field a uniform gradient of potential can be discerned by placing a charged and uncharged particle in the field - one remains stationary, the other is set into motion. An electromagnetic field of the form of a ripple must manifest itself as the second time derivative of the potential. In the case of gravity waves there exists no counterpart to the uncharged particle; thus, in order to detect a gravity field at all, recourse must be made to a second derivative of the potential and in order to detect a wave in a gravity field it is necessary, in effect, to measure the third derivative of the potential.

There is a conceptual difficulty also associated with gravity waves. Thus, a wave is usually pictured as a ripple superimposed on a steady motion. The nonlinearity of the field equations invalidates such a superposition procedure in the present instance. Reverting to the two-body problem, appeal to the electromagnetic analogy would lead us to believe that, if two bodies are gyrating about one another, they will constantly lose energy in the form of gravity waves and, as a result, come together and coalesce. As mentioned previously, there is no unanimity as to whether this does happen. It is of interest to note the magnitude of the effects under discussion. Thus, according to reference 24 for the two bodies pursuing circular paths about one another, the rate at which energy is lost in the form of gravity waves is given by the equation

$$\frac{dE}{dt} = -\frac{32G}{c^5}\left(\frac{m_1 m_2}{m_1 + m_2}\right)^2 r^4 \omega^6 \tag{69}$$

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where

G	universal gravitational constant
m_1, m_2	masses of two bodies
r	distance between bodies
ω	angular speed of rotation

As a result of this loss of energy, the distance between the masses constantly diminishes and is given by

$$\frac{dr}{dt} = -\frac{64G^3 m_1 m_2 (m_1 + m_2)}{5c^5 r^3} \tag{70}$$

If these equations are applied to the earth-sun system, the system appears to be losing energy at the rate of 10^{10} ergs per second and the resulting diminution in distance between the earth and the sun is of the order of 0.3 millimeter in 1,000 million years! In equation (69) note that ω appears to the sixth power. Can the effect be made sufficiently sizeable to be measureable by spinning at very high rates of rotation? In this connection note (ref. 25) that the Air Force was contemplating making an experiment in which they rotate a flywheel (4,000-pound mass) at 100,000 revolutions per minute and attempt to measure the rate at which the flywheel loses energy through the emission of gravitational waves.

$$\frac{dE}{dt} = -\frac{32G}{5c^5} I^2 \omega^6 \tag{71}$$

By inserting the figures quoted the rate of loss of energy is found to be about 4.054×10^{-14} ergs per second.

CONCLUDING REMARKS

Much of the data presented in this paper pertaining to the figure of the earth, its gravitational field, and other astronomical constants has been acquired by painstaking investigations, both experimental and theoretical, extending over a long period of time. Nevertheless, by virtue of the inherent inaccuracies in the data pertaining to the earth and even greater uncertainties involved in converting from astronomical to laboratory units such data does not provide a really adequate basis for the "precision" computation of satellite orbits and the trajectories of space probes. Within recent years application of newer techniques has improved the situation somewhat. Thus, observation of the perturbations to which earth satellites are subject has thrown a great deal of light on the fine structure of the earth's gravitational field, and presumably, once satellites can be established around the moon and other planetary bodies, equally valuable information can be obtained about them. Also, with the development of radar techniques to the point that discernible reflection signals can be obtained from Venus, it seems that really accurate evaluation of the astronomical unit in terms of kilometers will be shortly forthcoming. Not only will such data fill the practical need already alluded to but has intrinsic scientific interest. For example, knowledge of the fine structure of the terrestrial and lunar gravitational fields will throw light on the internal structure of these bodies which, in turn, has bearing on their manner of origin. Continued application of these techniques will clearly yield data of great value.

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The relativistic effects lie on the threshold of observation; hence, their omission in trajectory computations can scarcely be said to be a matter of practical concern. However, the advent of satellites does provide a potential capability of providing checks on the general theory of relativity by making experiments under idealized conditions which cannot be duplicated in earthbound laboratories.

Langley Research Center,
National Aeronautics and Space Administration,
Langley Station, Hampton, Va., March 23, 1962.

APPENDIX

SPHERICAL HARMONICS

It has been our aim throughout to keep the text as descriptive as possible and to deemphasize the mathematical aspects of the subject. However, from place to place some mathematics has intruded; thus, in considering the gravitational field of the earth, one could not avoid the introduction of the notation of spherical harmonics. For those readers who are unfamiliar with harmonic theory, this appendix has been included to provide an introduction to some of the ideas and concepts which underlie this mathematical technique. An attempt has been made to develop the theory by plausible argument rather than by rigorous reasoning. Those readers preferring the latter approach are referred to one of the more conventional mathematical texts. The following symbols are used in this appendix.

l, m, n	direction cosines
$P_n = P_n^{(0)}$	zonal harmonics (Legendre functions of the first kind of degree n)
$P_n^{(n)}$	sectorial harmonics
$P_n^{(m)}$	tesseral harmonics
r, θ, ϕ	polar coordinates of a point
S_n	rational integral spherical harmonics of degree n
V	denotes general harmonic function
x, y, z	Cartesian coordinates of a point
$\mu = \cos\theta$	

General Considerations

Some definitions and some elementary theorems are presented.

Any function satisfying Laplace's equation $\nabla^2 V = 0$ is said to be a harmonic function. In general, a solution of Laplace's equation is wholly lacking in periodicity and one may question therefore the appropriateness of attaching the name harmonic to such a solution.[17] A measure of justification is provided by the following considerations: just as an arbitrary function can be developed in terms of harmonic functions so the general solution of Laplace's equation can be - and, in general, will be - developed in terms of spherical surface harmonics, ellipsoidal surface harmonics, and so forth. Such functions merit the term harmonics since they do indeed undulate over their respective surfaces of definition, that is, sphere, ellipsoid, and so forth.

A harmonic function which is homogeneous[18] in xyz of integral[19] degree n is termed a spherical solid harmonic of degree n. By virtue of its homogeneity

$$\begin{aligned} V(xyz) &= V(r \sin\theta \cos\phi,\ r\sin\theta\sin\phi,\ r\cos\theta) \\ &= r^n V(\sin\theta\cos\phi,\ \sin\theta\sin\phi,\ \cos\theta) \\ &= r^n S_n(\theta\phi) \end{aligned}$$

$S_n(\theta\phi)$ is termed a spherical surface harmonic of degree n.

Substituting into Laplace's equation

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2} = 0 \qquad \text{(A1)}$$

it is found that S_n must satisfy the equation

$$\frac{\partial^2 S_n}{\partial\theta^2} + \cot\theta\frac{\partial S_n}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2 S_n}{\partial\phi^2} + n(n+1)S_n = 0 \qquad \text{(A2)}$$

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[17] Responsibility for any inadequacies of notation or terminology must be borne by Thomson and Tait, for it was in their "Treatise on Natural Philosophy" that the current notation and terminology was first introduced.

[18] A function $f(xyz)$ is homogeneous of degree n in xyz if

$$f(px,py,pz) = p^n f(xyz)$$

[19] Although n, in theory, can assume any value whatever, if n be nonintegral, the solution exhibits singular behavior along the polar axis. Such solutions must be precluded from potential theory since it is a requirement that external to the gravitating masses the potential must be everywhere well behaved.

and, if such be the case, it is readily verified that $r^{-n-1}S_n(\theta\phi)$ also satisfies Laplace's equation. The following theorems can be established:

Theorem 1: If $V(xyz)$ be a spherical solid harmonic of degree n, then $\frac{V(xyz)}{r^{2n+1}}$ is a spherical solid harmonic of degree $-(n + 1)$.

Theorem 2: If $V(xyz)$ be a spherical solid harmonic of degree n, then $\frac{\partial^{p+q+r}V}{\partial x^p \partial y^q \partial z^r}$ is a spherical solid harmonic of degree $n - p - q - r$.

Theorem 3: The value of any finite single-valued function of position on a sphere of unit radius can be expressed at every point at which the function is continuous as a series of rational integral harmonics, provided the function has only a finite number of lines and points of discontinuity and of maxima and minima on the surface.

Theorem 2 is almost self-evident if Laplace's equation is viewed in its Cartesian form.

Now $1/r$ defines the potential associated with unit mass at the origin. As such it must satisfy Laplace's equation and indeed is a spherical solid harmonic of degree -1. By theorem 2 $\frac{\partial}{\partial x}\left(\frac{1}{r}\right)$, $\frac{\partial}{\partial y}\left(\frac{1}{r}\right)$, and $\frac{\partial}{\partial z}\left(\frac{1}{r}\right)$ are spherical solid harmonics of degree -2. Whence, by virtue of the linearity of Laplace's equation,

$$l \frac{\partial}{\partial x}\left(\frac{1}{r}\right) + m \frac{\partial}{\partial y}\left(\frac{1}{r}\right) + n \frac{\partial}{\partial z}\left(\frac{1}{r}\right) = \frac{\partial}{\partial h}\left(\frac{1}{r}\right)$$

(where $\partial/\partial h$ implies differentiation in a direction having direction cosines l, m, and n) is also a spherical solid harmonic of degree -2. This result is capable of immediate generalization; thus, $\frac{\partial^n}{\partial h_1 \partial h_2 \ldots \partial h_n}\left(\frac{1}{r}\right)$ is a spherical solid harmonic of degree $-n - 1$.

Homogeneous solutions of Laplace's equation can themselves assume a multivariety of form. Thus $\tan^{-1}\frac{y}{r}$; $\log_e \frac{r + z}{r - z}$; $\tan^{-1}\frac{y}{r} \log_e \frac{r + z}{r - z}$ are homogeneous solutions of degree zero. An especially important subclass of homogeneous solutions are the so-called rational integral harmonics. These are homogeneous solutions of Laplace's equation which are also rational integral (or polynomial functions). The general homogeneous rational integral function of degree n takes the form

$$
\begin{aligned}
a_{0,0}x^n &+ a_{1,0}x^{n-1}y + a_{2,0}x^{n-2}y^2 + \cdots + a_{n,0}y^n \\
&+ a_{0,1}x^{n-1}z + a_{1,1}x^{n-2}yz + \cdots + a_{n-1,1}y^{n-1}z \\
&\qquad + a_{0,2}x^{n-2}z^2 + \cdots + a_{n-2,2}y^{n-2}z^2 \\
&\qquad\qquad \vdots \\
&\qquad\qquad + a_{0n}z^n
\end{aligned}
$$

The total number of terms is $1 + 2 + 3 + \cdots + (n + 1) = \frac{1}{2}(n + 1)(n + 2)$.

If this function is substituted into Laplace's equation, the following equation is obtained:

$$
\begin{aligned}
b_{0,0}x^{n-2} &+ b_{1,0}x^{n-3}y + \cdots + b_{n-2,0}y^{n-2} \\
&+ b_{0,1}x^{n-3}z + \cdots + b_{n-3,1}y^{n-3}z \\
&\qquad \vdots \\
&\qquad + b_{0,n-2}z^{n-2} = 0
\end{aligned}
$$

Since the terms appearing on the left-hand side are linearly independent if the equation is to be satisfied, the individual coefficients $\left(\text{of which there are } \frac{1}{2}n(n - 1)\right)$ must be zero. This condition places $\frac{1}{2}n(n - 1)$ requirements on our original $\frac{1}{2}(n + 1)(n + 2)$ coefficients from which it follows that there are $2n + 1$ linearly independent rational integral harmonics of degree n.

These rational integral harmonics owe their importance to theorem 3.

Thus, consider the general Dirichlet problem of spherical harmonics. After having specified a function $f(\theta\phi)$ over the surface of unit sphere to obtain a solution to Laplace's equation which converges to the function $f(\theta\phi)$ as the unit sphere is approached.

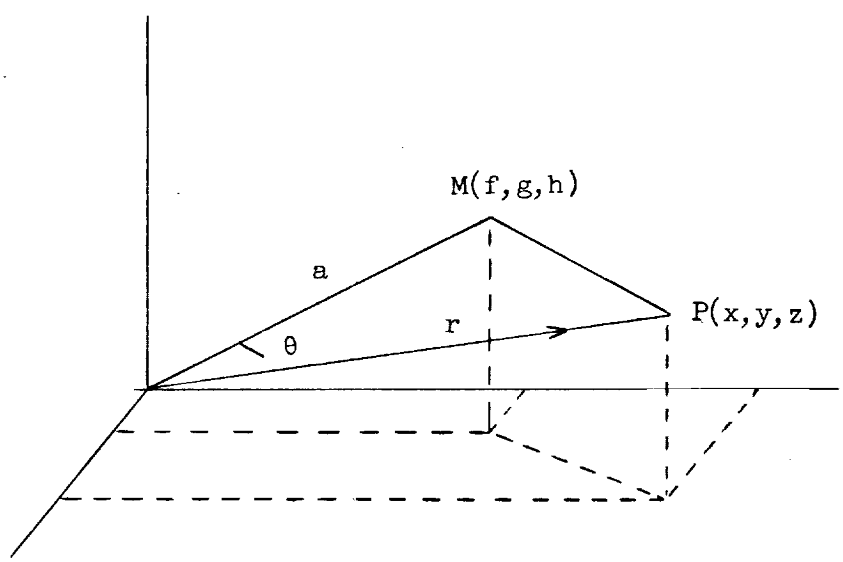
By theorem 3, $f(xyz) = a_0S_0(\theta\phi) + a_1S_1(\theta\phi) + a_2S_2(\theta\phi) + \ldots$ where S_0, S_1, S_2, and so forth are rational integral harmonics of degrees 0, 1, 2, and so forth, then the solution desired can be written down immediately

$$\frac{a_0S_0(\theta\phi)}{r} + \frac{a_1S_1(\theta\phi)}{r^2} + \frac{a_2S_2(\theta\phi)}{r^3} + \ldots$$

By theorem 2, to each rational integral harmonic of degree n there is a rational integral harmonic of degree $-(n + 1)$. It follows, therefore, that there are $2n + 1$ rational integral harmonics of degree $-(n + 1)$. Specifically setting $n = 0$ there is only one independent rational integral harmonic of degree -1, that is, $1/r$. By successive differentiation of $1/r$ in all possible directions the entire system of rational integral harmonic functions can be developed.

Field of a Single Mass Point

The potential associated with a unit mass at M (see sketch 44) will now be considered.



Sketch 44

The potential at P, due to unit mass at M, is given by

$$V_P = \frac{1}{MP} = \frac{1}{\left[(x - f)^2 + (y - g)^2 + (z - h)^2\right]^{1/2}} \tag{A3}$$

Expansion in a Taylor series in the coordinates f, g, and h yields

$$V_P = \frac{1}{r} - \left(f \frac{\partial}{\partial x} + g \frac{\partial}{\partial y} + h \frac{\partial}{\partial z}\right)\left(\frac{1}{r}\right) + \cdot \cdot \cdot$$

$$+ \frac{(-)^n}{\lfloor\underline{n}}\left(f \frac{\partial}{\partial x} + g \frac{\partial}{\partial y} + h \frac{\partial}{\partial z}\right)^n\left(\frac{1}{r}\right) + \cdot \cdot \cdot \tag{A4}$$

which is valid everywhere external to the sphere embracing the mathematical singularity (point mass), that is, $r > a$.[20]

The terms of the Taylor series can be rewritten thus

$$\left(f \frac{\partial}{\partial x} + g \frac{\partial}{\partial y} + h \frac{\partial}{\partial z}\right)\left(\frac{1}{r}\right) = a \frac{\partial}{\partial h}\left(\frac{1}{r}\right)$$

$$\left(f \frac{\partial}{\partial x} + g \frac{\partial}{\partial y} + h \frac{\partial}{\partial z}\right)^n\left(\frac{1}{r}\right) = a^n \frac{\partial^n}{\partial h^n}\left(\frac{1}{r}\right)$$

Thus the Taylor series becomes

$$V_P = \frac{1}{r} - a \frac{\partial}{\partial h}\left(\frac{1}{r}\right) + \cdot \cdot \cdot \cdot + \frac{(-)^n a^n}{\lfloor\underline{n}} \frac{\partial^n}{\partial h^n}\left(\frac{1}{r}\right) + \cdot \cdot \cdot \tag{A5}$$

In this form the series is amenable to physical interpretation. Thus $1/r$ defines the potential associated with a point source at the origin; $\frac{\partial}{\partial h}\left(\frac{1}{r}\right)$ defines the potential associated with a dipole at the origin with its axis in the direction OM; $\frac{\partial^2}{\partial h^2}\left(\frac{1}{r}\right)$ defines the potential associated with a quadripole at the origin with its axis in the direction OM; and so on.

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[20]Note the similarity with the situation in the complex plane where the expansion of a function in terms of inverse powers of z is valid everywhere external to the circle embracing the singularities of the function in question.

Alternative series developments for the potential can be obtained as follows:

$$V_P = \frac{1}{MP} = \frac{1}{\left(r^2 + a^2 - 2ar\cos\theta\right)^{1/2}}$$

$$= \frac{1}{r}\left[1 + \frac{a}{r}P_1(\mu) + \ldots + \frac{a^n}{r^n}P_n(\mu) + \ldots\right] \qquad (r > a) \qquad \text{(A6)}$$

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$$V_P = \frac{1}{a}\left[1 + \frac{r}{a}P_1(\mu) + \ldots + \frac{r^n}{a^n}P_n(\mu) + \ldots\right] \qquad (r < a) \qquad \text{(A7)}$$

where

$$\mu = \cos\theta$$

$$P_1(\mu) = \mu$$

$$P_2(\mu) = \frac{1}{2}\left(3\mu^2 - 1\right)$$

$$P_3(\mu) = \frac{1}{2}\left(5\mu^3 - 3\mu\right)$$

$$\vdots$$

These relations are collectively referred to as "Legendre functions of the first kind." Series (A4) and (A6) are entirely equivalent. One is permitted therefore to equate corresponding terms

$$\frac{a^n}{r^{n+1}}P_n(\mu) = \frac{(-)^n a^n}{\lfloor\underline{n}}\frac{\partial^n}{\partial h^n}\left(\frac{1}{r}\right) \qquad \text{(A8)}$$

From theorem 2 it is known that the right-hand side of equation (A8) defines a spherical harmonic function of degree $-n - 1$; thus, $P_n(\mu)$ is a spherical surface harmonic. As such, it satisfies equation (A2) or rather the equation derivable from equation (A2) when the lack of dependence of P_n on the coordinate ϕ is considered

$$\frac{d}{d\mu}\left[\left(1 - \mu^2\right)\frac{dP_n}{d\mu}\right] + n(n + 1)P_n = 0 \qquad \text{(A9)}$$

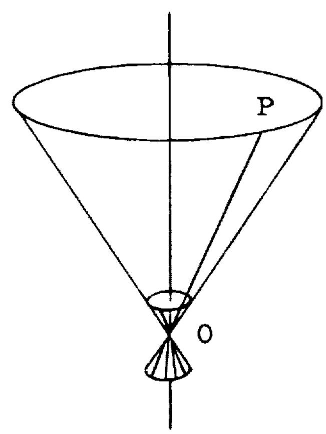
This is a second-order differential equation and, as such, it possesses a solution which is linearly independent of $P_n(\mu)$. However, this additional solution exhibits singular behavior along the polar axis and, as such, can play no role in potential theory.

From equation (A8) it is seen that P_0, P_1, P_2, . . . define the spatial distribution of potential associated with a source, dipole, quadripole, and so forth. Indeed, by using equation (A8) the geometric form of the singularities in question can be readily established. The results are presented for the first four singularities in table IV.

Consider the singularity associated with $P_3(\mu)$. It has no net magnitude, the positives canceling the negatives (in a sense it might be said to have no zeroeth moment). The first and second moments are identically zero. The fourth and higher moments all tend to zero in the limit by virtue of the fact that $a \to 0$ and $m \to \infty$ such that ma^3 is finite. In other words, $P_3(\mu)$ contributes only to the third moment and in general $P_n(\mu)$ contributes only to the nth moment.

Gravitational Field of Axisymmetric Mass Distribution

The gravitational field of an axisymmetric mass distribution can be regarded as made up of uniform rings having their axes coinciding with the axis of symmetry. Consider one such ring (sketch 45). The



Sketch 45

potential associated with the particle at P can be represented, external to a sphere embracing the entire mass distribution, by a source at O plus a dipole, quadripole, and so forth at O, all with their axes alined with OP. To each particle of the ring there corresponds an equivalent singularity distribution. Consider for the moment the dipoles. They are of uniform strength; however, their axes are distributed over the conical surface at 0. Such a distribution is clearly equivalent to a dipole alined with the axis of symmetry. By the same token the conical distribution of quadripoles can be replaced by a single quadripole along the axis of symmetry. Thus the potential associated with an axisymmetric mass distribution can be represented by singularities located at the origin and having their axes coincident with the axis of symmetry. In other words, an axisymmetric mass distribution can be expanded as a series in $P_0(\mu)$, $P_1(\mu)$, $P_2(\mu)$, . . . where P_0 is a measure of total mass, P_1 is a measure of the first moment of the mass distribution, P_2 is a measure of the second moment of the mass distribution, and so on. Now these successive moments are entirely separate and unrelated characteristics of the mass distribution. (This fact is reflected in their having different dimensions.) Thus P_0, P_1, P_2, . . . are entirely unrelated. In mathematical parlance therefore P_0, P_1, P_2, . . . are said to be mutually orthogonal

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$$\int P_m P_n \, dw = 0 \qquad (m \neq n) \qquad \text{(A10)}$$

It can be shown that, if an arbitrary axisymmetric function be defined over the unit sphere and this function be approximated by a series of Legendre functions, then by virtue of equation (A10) a least-square fit to the function in question will be obtained in which equal weight is ascribed to all points of the spherical surface. This property can be used as a means of defining the Legendre functions. (See ref. 26.) Its significance in gravimetric work in which observations are made at a number of discrete points and the least-square fit applied to the data is obvious.

Discussion of Two-Dimensional Case

Before proceeding to a discussion of the case of a nonaxisymmetric mass distribution, it is instructive at this juncture to discuss the two-dimensional case. It is shown that in a very real sense the theory of spherical harmonics is an immediate generalization of Fourier's harmonic analysis in two dimensions.

Laplace's equation in polar coordinates in two dimensions takes the form

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2} = 0 \qquad \text{(A11)}$$

If homogeneous solutions of degree n $\left(V = r^n S_n(\theta)\right)$ are sought, then S_n satisfies the equation[21]

$$\frac{d^2 S_n}{d\theta^2} + n^2 S_n = 0 \tag{A12}$$

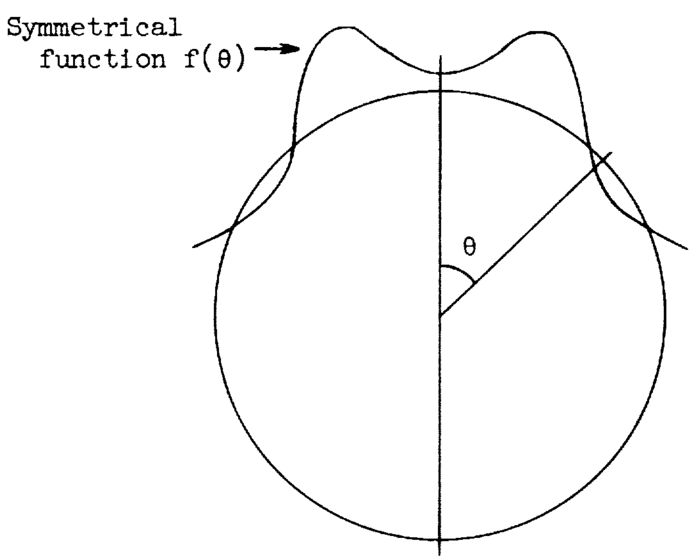
Thus

$$S_n = \cos n\theta$$

or

$$S_n = \sin n\theta$$

These functions are the rational integral solutions in two dimensions. Fourier's theorem can be formulated in strict conformity with theorem 3, thusly, the value of any function of position on the circumference of a circle can be expressed at every point of the circumference at which the function is continuous in terms of rational integral function (circular functions of multiple angles) provided the function is single valued and has only a finite number of discontinuities and of maxima and minima on the circumference of the circle. (See sketch 46.)



Sketch 46

[21]To ensure periodicity with respect to θ, n must be integral.

If a requirement of symmetry with respect to a diameter is imposed, an otherwise arbitrary function can be expanded as a cosine series

$$f(\theta) = a_0 + a_1 \cos \theta + a_2 \cos 2\theta + a_3 \cos 3\theta + . . . \tag{A13}$$

where 1, $\cos \theta$, $\cos 2\theta$, $\cos 3\theta$, . . . are strictly analogous to P_0, P_1, P_2, P_3, ... Just as the geometric form of the singularities associated with the Legendre functions could be established, those associated with circular functions can be determined. The results are presented in table V.

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It is possible, if desired, to develop the theory of circular and elliptic harmonics, and so forth, strictly analogously to the theory of spherical and ellipsoidal harmonics, and so forth. In practice, however, this is not done because in two dimensions there is a much more powerful tool for the solution of potential problems in the guise of the theory of analytic functions.

General Nonsymmetric Case

In this instance a general solution of the equation

$$\frac{\partial^2 S_n}{\partial \theta^2} + \cot \theta \frac{\partial S_n}{\partial \theta} + \frac{1}{\sin^2\theta} \frac{\partial^2 S_n}{\partial \phi^2} + n(n + 1)S_n = 0 \tag{A14}$$

is required. Separating variables and setting $S_n(\theta\phi) = \Theta(\theta)\Phi(\phi)$ yields

$$\Phi \frac{d^2\Phi}{d\phi^2} = c \tag{A15}$$

$$\frac{\sin^2\theta}{\Theta}\left[\frac{d\ \Theta}{d\theta^2} + \cot \theta \frac{d\Theta}{d\theta} + n(n + 1)\Theta\right] = -c \tag{A16}$$

To ensure periodicity with respect to the coordinate ϕ, the constant c appearing in equation (A15) must be of the form $c = -p^2$ where p is integral. Hence equation (A16) assumes the form

$$\frac{d}{d\mu}\left[\left(1 - \mu^2\right)\frac{d\Theta}{d\mu}\right] + \left[n(n + 1) - \frac{p^2}{1 - \mu^2}\right]\Theta = 0 \tag{A17}$$

Moreover, since it is required that Θ be a periodic function of θ, this demands that $p \leqq n$. Thus, the permissible solutions of degree n are $P_n^{(0)}$, $P_n^{(1)}\cos \phi$, $P_n^{(1)}\sin \phi$, $P^{(2)}\cos 2\phi$, $P_n^{(2)}\sin 2\phi$, . . . $P_n^{(n)}\cos n\phi$, $P_n^{(n)}\sin n\phi$. These constitute $2n + 1$ linearly independent rational integral harmonic functions which by the manner of their derivation are mutually orthogonal and $P_n^{(0)}$, $P_n^{(1)}$, . . . $P_n^{(n)}$ are termed associated Legendre functions of the first kind.

What is the nature of the singularities associated with these generalized harmonic functions? The composite form of the functions $P_n^{(k)}\cos k\phi$ suggest that the singularities are synthesized out of the axial singularities of table IV and the star-shaped singularities of table V. This indeed proves to be the case; thus, $P_n^{(k)}\cos k\phi$ is obtained by differentiating the star-shaped singularity associated with $\cos k\phi$ $n - k$ times in the Z-direction. A representation of the singularity associated with $P_4{}^3\cos 3\phi$ is given in sketch 47.

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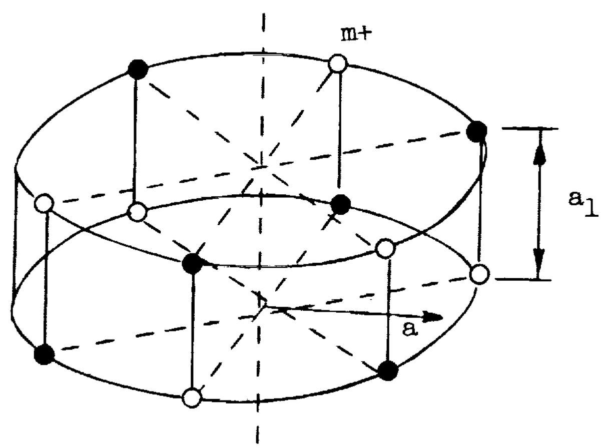
Nature of singularity associated with $P_4{}^3\cos 3\phi$

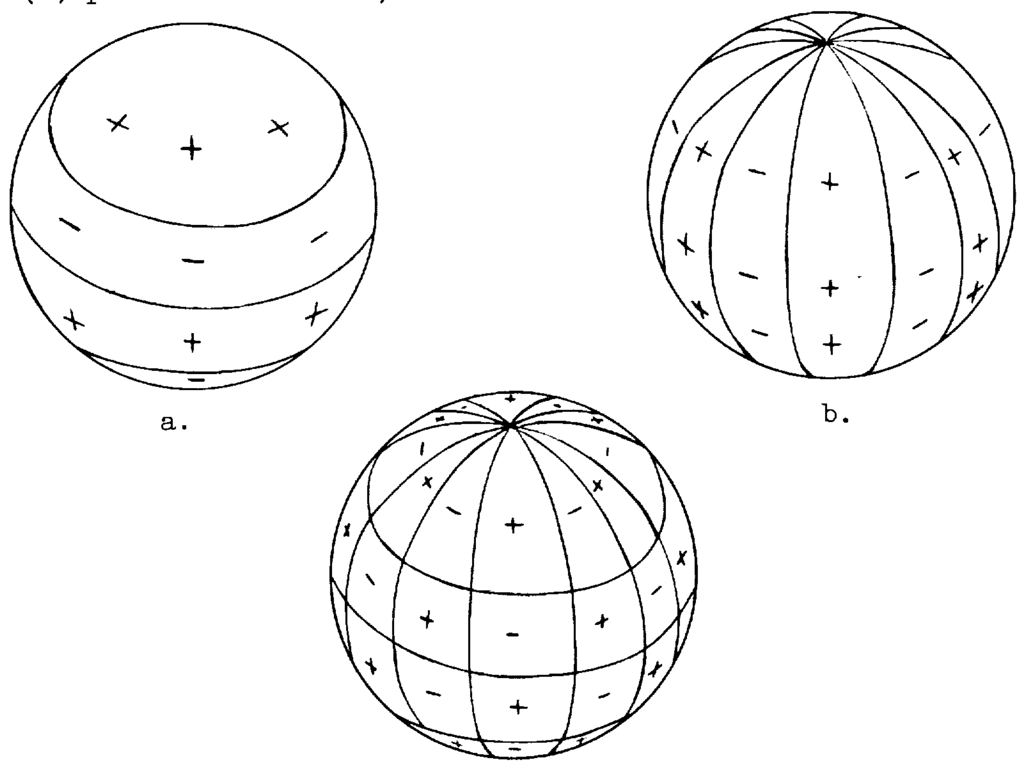


ma^4 is finite

Sketch 47

The functions $P_n^{(0)}$ are the so-called zonal harmonics discussed previously; they alternate in sign, the zero lines subdividing the sphere into equal zonal segments. (See (a) part of sketch 48.) The functions $P_n^{(n)}\cos n\phi$ and $P_n^{(n)}\sin n\phi$ are termed sectorial harmonics; the zero lines in this case divide the sphere up into equal sectors. (See

(b) part of sketch 48.) The intervening harmonics are called tesseral harmonics by virtue of their dividing the sphere into equal tesserae. (See (c) part of sketch 48.)



Sketch 48

If a function, defined on the surface of unit sphere, be approximated by a series of harmonic functions, by virtue of the orthogonality of the latter, the series will provide a least-square fit in which equal weights are given to all points of the spherical surface. In this respect the harmonic functions are singularly appropriate for series representation of a function, the value of which has been measured at a finite number of distinct points.

Generalization of Concepts

Let us suppose a function f is defined along the line segment between -1 and 1. Imagine a unit sphere to be centered on the point 0 and the function f to be projected radially outwards onto the surface of the sphere. Thus, the value of f at distance x is assigned to all points along the parallel of latitude of this distance from the equatorial plane. The function thus defined over the surface of unit sphere can be represented by a series of zonal harmonics. Since this series attaches equal weight to equal zonal segments of the sphere and since equal zonal segments correspond to equal segments of the linear interval -1 to 1, it follows that the expansion in terms of Legendre functions will uniformly approximate the function over the entire interval.

If, on the other hand, the unit circle was centered on 0 and the function f projected onto the circumference of the circle in a manner strictly analogous to that described above, the function defined on the circle could be expanded as a cosine series. In this case, however, the expansion gives equal weights to equal arc lengths; however, equal arc lengths no longer correspond to equal lengths of the diameter and the expansion of f between -1 and 1 in terms of circular functions gives greater weight to the extremities of the interval than to the center.

The circular functions and the Legendre functions could have been obtained by seeking sets of functions which are orthogonal and which provide uniform least-square fit to functions which are defined on the circle and sphere, respectively. This would provide immediately a method of generalizing to higher dimensions. Orthogonal polynomials to provide uniform fit to functions defined over the surfaces of hyperspheres of four and higher dimensions are sought. In this way the Gegenbauer polynomials can be obtained. Such polynomials can, of course, be used to approximate a function over the interval -1 to 1. These approximations will, if so applied, have associated with them certain characteristic weighting functions.

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TABLE I.- DIMENSIONS OF THE EARTH ELLIPSOID

FROM ARC MEASUREMENTS

[Taken from ref. 1 (p. 230)]

Author:	Year	r_q	$1/\epsilon$
Bouguer, Maupertuis	1738	6,397,300	216.8
Delambre	1800	6,375,653	334.0
Walbeck	1819	6,376,896	302.8
Everest	1830	6,377,276	300.8
Airy	1830	6,376,542	299.3
Bessel	1841	6,377,397	299.15
Clarke	1857	6,378,345	294.26
Pratt	1863	6,378,245	295.3
Clarke	1866	6,378,206	295.0
Clarke	1880	6,378,249	293.5
Bonsdorff, A.	1888	6,378,444	298.6
Hayford	1906	6,378,283	297.8
Helmert	1907	6,378,200	298.6
Hayford	1910	6,378,388	297.0
Heiskanen	1926	6,378,397	(279.0)
Krassowski	1938	6,378,245	298.3
Jeffreys	1948	6,378,099	297.1
Ledersteger	1951	6,378,298	(297.0)
U.S. Army Map Service (Hough)	1956	6,378,260	(297.0)

TABLE II.- GRAVITY FORMULAS IGNORING LONGITUDINAL VARIATIONS

[Taken from ref. 1 (p. 78)]

Helmert	1901	$978.030(1 + 0.005302 \sin^2\phi - 0.000007 \sin^2 2\phi)$ $a = 0.0033535 = \frac{1}{298.2}$
Bowie	1917	$978.039(1 + 0.005294 \sin^2\phi - 0.000007 \sin^2 2\phi)$ $a = 0.0033614 = \frac{1}{297.5}$
Heiskanen	1928	$978.049(1 + 0.005289 \sin^2\phi - 0.000007 \sin^2 2\phi)$ $a = 0.0033664 = \frac{1}{297.06}$
Heiskanen, Uotila	1957	$978.0496(1 + 0.0052934 \sin^2\phi - 0.0000059 \sin^2 2\phi)$ $a = 0.0023622 = \frac{1}{297.4}$
International	1930	$978.0490(1 + 0.0052884 \sin^2\phi - 0.0000059 \sin^2 2\phi)$ $a = 0.0033670 = \frac{1}{297.00}$

TABLE III.- GRAVITY FORMULAS INCLUDING LONGITUDINAL VARIATIONS

[Taken from ref. 1 (p. 79)]

Helmert	1915	$978.052\left[1 + 0.005285 \sin^2\phi - 0.0000070 \sin^2 2\phi + 0.000018 \cos^2\phi \cos^2(\lambda + 17^\circ)\right]$
Heiskanen	1924	$978.052\left[1 + 0.005285 \sin^2\phi - 0.0000070 \sin^2 2\phi + 0.000027 \cos^2\phi \cos 2(\lambda - 18^\circ)\right]$
Heiskanen	1928	$978.049\left[1 + 0.005293 \sin^2\phi - 0.0000070 \sin^2 2\phi + 0.000019 \cos^2\phi \cos 2(\lambda - 0^\circ)\right]$
Niskanen	1945	$978.0468\left[1 + 0.0052978 \sin^2\phi - 0.0000059 \sin^2 2\phi + 0.0000230 \cos^2\phi \cos 2(\lambda + 4^\circ)\right]$
Uotila	1957	$978.0516\left[1 + 0.0052910 \sin^2\phi - 0.0000059 \sin^2 2\phi + 0.0000106 \cos^2\phi \cos 2(\lambda + 6^\circ)\right]$

TABLE IV.- FORM OF SINGULARITIES USED IN

SPHERICAL HARMONIC ANALYSIS

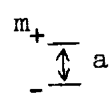
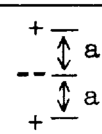
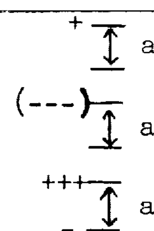
Legendre function	Radial fall off in intensity	Geometric form of singularity	$a \rightarrow 0$ and $m \rightarrow \infty$ so that -
$P_0(\mu)$	$\frac{1}{r}$	+	
$P_1(\mu)$	$\frac{1}{r^2}$		ma finite
$P_2(\mu)$	$\frac{1}{r^3}$		ma^2 finite
$P_3(\mu)$	$\frac{1}{r^4}$		m^3 finite

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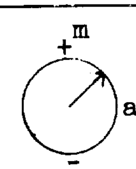
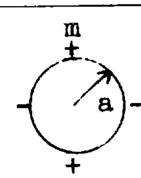
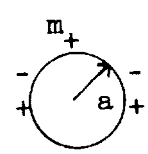
TABLE V.- FORM OF SINGULARITIES USED IN FOURIER ANALYSIS

Cosine circular functions	Radial fall off in intensity	Geometric form of singularity	$a \to 0$ and $m \to \infty$ so that -
1	$\log_e$	+	
$\cos \theta$	$\frac{1}{r}$		ma^2 finite
$\cos 2\theta$	$\frac{1}{r^2}$		ma^2 finite
$\cos 3\theta$	$\frac{1}{r^3}$		ma^3 finite

NASA TN D-1270
National Aeronautics and Space Administration.
THE GRAVITATIONAL FIELD ENVIRONMENT OF AN EARTH SATELLITE. David Adamson. August 1962. 115p. OTS price, $2.50.
(NASA TECHNICAL NOTE D-1270)

This report surveys what is currently known about the shape of the earth and the gravitational fields of the earth, the sun, and the moon. The various techniques used to obtain these data are described. A brief survey of the relativistic effects on orbits has also been included.

I. Adamson, David
II. NASA TN D-1270

(Initial NASA distribution:
6, Astronomy;
21, Geophysics and geodesy;
33, Physics, theoretical;
46, Space mechanics;
47, Satellites.)

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